Flow analysis from cumulants: a practical guide

Nicolas Borghini,[1] Phuong Mai Dinh,[2] and Jean-Yves Ollitrault[2,3]

[1] *Service de Physique Théorique, CP225, Université Libre de Bruxelles, 1050 Brussels, Belgium*
[2] *Service de Physique Théorique, CEA-Saclay, 91191 Gif-sur-Yvette cedex, France*
[3] *L.P.N.H.E., Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris cedex 05, France*

We have recently proposed a new method of flow analysis, based on a cumulant expansion of multiparticle azimuthal correlations. Here, we describe the practical implementation of the method. The major improvement over traditional methods is that the cumulant expansion eliminates order by order correlations not due to flow, which are often large but usually neglected.

I. INTRODUCTION

The measurement of the azimuthal distributions of outgoing particles with respect to the reaction plane in noncentral heavy ion collisions—the flow analysis—is an important probe of the interaction region of the collision [1]. In particular, it has raised much interest at ultrarelativistic energies [2] where it may signal the formation of a quark-gluon plasma [3]. In addition, combining flow and two-particle interferometry results yields a three-dimensional picture of the emitting source [4,5]. Therefore, accurate flow measurements are highly needed.

Azimuthal distributions are characterized by the Fourier coefficients [6]

$$v_n \equiv \left\langle e^{in(\phi-\Phi_R)} \right\rangle = \langle \cos n(\phi - \Phi_R) \rangle , \tag{1}$$

where ϕ is the azimuthal angle of an emitted particle in the laboratory frame, Φ_R is the azimuth of the reaction plane, and angular brackets denote a statistical average over many particles and events. Ideally, v_n should be measured for various particles as a function of their transverse momentum p_T and rapidity y ("differential" flow). The first two harmonics v_1 and v_2 are the so-called directed and elliptic flows, respectively.

Since the azimuth Φ_R in a given event is unknown, the coefficients v_n are extracted from the azimuthal correlations between outgoing particles. The underlying idea is that the correlation of every particle with the reaction plane induces correlations between the particles. Standard methods extract flow from two-particle azimuthal correlations, either directly [7], or through the correlation between two "subevents" [8,9]. However, the correlation between two given particles is not only due to flow, and the other sources of correlation—as, e.g., quantum Bose-Einstein effects, momentum conservation, resonance decays, jets—may dominate the measured signal, especially for peripheral collisions, thereby spoiling the validity of the analysis [10,11]. The impact of "nonflow" correlations on the flow analysis might tentatively be minimized: cuts in phase space can be used to avoid the influence of quantum effects and resonance decays, while the contribution of momentum conservation to the measured correlation can be calculated and subtracted [11,12]. However, the various recipes require some *a priori* knowledge of nonflow correlations; furthermore it is necessary to assume that *all* sources of such correlations are known and accounted for, which may not be true.

To remedy the contamination from nonflow correlations, which amounts to systematic uncertainties on the flow values, we have introduced new methods of flow analysis, based on a cumulant expansion of multiparticle azimuthal correlations [13,14]. The principle of the methods is that when cumulants of higher order are considered, the relative contribution of nonflow effects, and thus the corresponding systematic error, decreases. Recently, this cumulant expansion has been successfully applied by the STAR Collaboration at RHIC [2,15].

More precisely, the cumulant of $2k$-particle azimuthal correlations, which we denote $c_n\{2k\}$ (where n is the Fourier harmonic and $2k$ is an even integer, in practice 2, 4 or 6), is a quantity built with all the measured azimuthal correlations up to order $2k$, i.e., the $\langle \exp[in(\phi_1 + \phi_2 + \cdots + \phi_{k'} - \phi_{k'+1} - \cdots - \phi_{k'+k''})] \rangle$, with $k' + k'' \leq 2k$. The key feature of the cumulant is that it eliminates the contribution of lower order correlations, so that only the genuine $2k$-particle correlation remains. Flow, which is essentially a collective effect, gives a contribution to the cumulant proportional to v_n^{2k}. The remaining contribution, from $2k$-particle nonflow correlations, scales as N^{1-2k}, where N is the total multiplicity of particles emitted in an event [13]. Therefore, the flow dominates if

$$v_n^{2k} \gg \frac{1}{N^{2k-1}} \qquad \Leftrightarrow \qquad v_n \gg \frac{1}{N^{1-1/2k}}. \tag{2}$$

Stated differently, if the cumulant $c_n\{2k\}$ is much larger than N^{1-2k}, it is dominated by flow, and therefore yields an estimate of v_n, which we denote $v_n\{2k\}$, with a systematic error (due to unknown nonflow correlations) of order $\mathcal{O}((Nv_n)^{1-2k})$. When k increases, the systematic error decreases: this is why the estimate $v_n\{4\}$ derived from the fourth cumulant is *a priori* more accurate than $v_n\{2\}$, which is in fact the value given by two-particle methods.

In the following, we describe the practical implementation of the method, referring the reader to Refs. [13,14] for further theoretical justifications. The first step consists in deriving a global measurement of v_n, integrated over some phase-space region, typically a detector acceptance, for a given centrality class (Sec. II): this is equivalent to reconstructing the reaction plane and obtaining the "event plane resolution" in the subevent method [8]. This "integrated" flow serves as reference for the differential flow analysis discussed in Sec. III. An important feature of our method is that it automatically takes into account azimuthal inhomogeneities in the acceptance of the detector. In the case of a detector with only partial azimuthal coverage, minor modifications occur, which are given in Sec. IV.

II. INTEGRATED FLOW

Consider a data set of N_{evts} events of approximately the same centrality, recorded in a run with a constant detector acceptance. In this section, we explain how estimates of the flow, integrated over the detector acceptance, can be obtained from cumulants of 2-, 4-, and 6-particle correlations.

We denote by ϕ_j the azimuths of the outgoing particles with respect to a fixed direction in the laboratory. The various quantities of interest are constructed from the real-valued generating function [14]

$$G_n(z) = \prod_{j=1}^{M} \left[1 + \frac{w_j}{M} \left(z^* e^{in\phi_j} + z e^{-in\phi_j} \right) \right] = \prod_{j=1}^{M} \left[1 + \frac{w_j}{M} \left(2x \cos(n\phi_j) + 2y \sin(n\phi_j) \right) \right], \tag{3}$$

where the product runs over M particles detected in a single event and $z = x + iy$ is an arbitrary complex number. This generating function has no physical meaning in itself, but after averaging over events, the coefficients of its expansion in powers of z and $z^* \equiv x - iy$ yield multiparticle azimuthal correlations of arbitrary orders. In practice, the variable z corresponds to interpolation points used to estimate the various quantities encountered in the analysis, as will be explained shortly.

As in the standard flow analysis, a weight w_j is attributed to particle j, which is a function of particle type, transverse momentum, and rapidity. Naturally, the integrated flow obtained from this generating function will be weighted by w, i.e. $V_n \equiv \langle w\, e^{in(\phi-\Phi_R)} \rangle$. The weight must be chosen so as to maximize the effects of flow relative to statistical fluctuations. As we shall see below, this is achieved by maximizing the dimensionless quantity $\chi_n \equiv V_n \sqrt{M/\langle w^2 \rangle}$ (this quantity also characterizes the event plane resolution in the standard flow analysis [9]). As a consequence, the optimal weight for a given particle is its flow $v_n(p_T, y)$ itself [13]. Thus, a thorough flow analysis should go twice through Secs. II and III. The first time, integrated flow can be extracted using some reasonable guess for the weights, thereby obtaining values for $v_n(p_T, y)$; in turn, these values will serve as weights in the second, final analysis [16].

In Eq. (3), the number M of particles should be the same for all events: in each event, a set of M particles must be randomly chosen out of the M_{tot} detected particles.

In order to obtain the cumulants, one first averages $G_n(z)$ over events, which yields an average generating function $\langle G_n(z) \rangle$. We then define [14]

$$\mathcal{C}_n(z) \equiv M \left[\langle G_n(z) \rangle^{1/M} - 1 \right]. \tag{4}$$

The cumulant of $2k$-particle correlations $c_n\{2k\}$ is the coefficient of $z^k z^{*k}/(k!)^2$ in the power-series expansion of $\mathcal{C}_n(z)$. To construct the first three cumulants, one may truncate the series to order $|z|^6$ and compute $\mathcal{C}_n(z)$ at the following interpolation points:

$$z_{p,q} = x_{p,q} + iy_{p,q}, \qquad x_{p,q} \equiv r_0 \sqrt{p} \cos\left(\frac{2q\pi}{q_{\max}} \right), \qquad y_{p,q} \equiv r_0 \sqrt{p} \sin\left(\frac{2q\pi}{q_{\max}} \right), \tag{5}$$

for $p = 1, 2, 3$ and $q = 0, \ldots, q_{\max} - 1$, where $q_{\max} \geq 8$. The parameter r_0 must be chosen as a compromise between errors due to higher order terms in the power-series expansion, which rapidly increase with r_0, and numerical errors. Assuming that the numerical error is proportional to the total number of elementary operations performed (of order MN_{evts}), we obtain the estimate $r_0 \simeq (\epsilon N_{\mathrm{evts}}^{3/2} M)^{1/8} \sqrt{M/\langle w^2 \rangle}$ where ϵ is the accuracy of elementary operations, typically 10^{-16} in double precision. This gives $r_0 \sim 2$ with weights of order unity and standard values of the multiplicity ($M \sim 300$) and number of events ($N_{\mathrm{evts}} \sim 20000$).

Once the values $\mathcal{C}_n(z_{p,q})$ have been computed, they must be averaged over the phase of z:

$$C_p \equiv \frac{1}{q_{\max}} \sum_{q=0}^{q_{\max}-1} \mathcal{C}_n(z_{p,q}), \qquad p = 1, 2, 3. \tag{6}$$

2

The cumulants of 2-, 4- and 6-particle correlations are then given respectively by

$$c_n\{2\} = \frac{1}{r_0^2}\left(3\,C_1 - \frac{3}{2}C_2 + \frac{1}{3}C_3\right), \qquad c_n\{4\} = \frac{2}{r_0^4}\left(-5\,C_1 + 4\,C_2 - C_3\right), \qquad c_n\{6\} = \frac{6}{r_0^6}\left(3\,C_1 - 3\,C_2 + C_3\right). \quad (7)$$

These cumulants are related to the weighted integrated flow $V_n \equiv \langle we^{in(\phi-\Phi_R)}\rangle$. This is the point where acceptance considerations come into play. If the detector has full azimuthal coverage, each cumulant $c_n\{2k\}$ gives an estimate of the corresponding V_n, which we denote by $V_n\{2k\}$:

$$V_n\{2\}^2 = c_n\{2\}, \qquad V_n\{4\}^4 = -c_n\{4\}, \qquad V_n\{6\}^6 = c_n\{6\}/4. \quad (8)$$

Generalized relations valid for detectors with partial azimuthal coverage or efficiency are given in Sec. IV.

In practice, the use of higher order cumulants is often limited by statistics. The order of magnitude of statistical errors can easily be estimated. The computation of the cumulant $c_n\{2k\}$ relies on the choice of $2k$ particles among M in each of the N_{evts} available events, i.e., it involves roughly $M^{2k}N_{\text{evts}}$ $(2k)$-uplets of particles. Taking into account the weights w, the resulting statistical uncertainty on $c_n\{2k\}$ is of order $\mathcal{O}(\langle w^2\rangle^k/\sqrt{M^{2k}N_{\text{evts}}})$. The relative error on $V_n\{2k\}$ is thus of order

$$\frac{\delta V_n\{2k\}}{V_n\{2k\}} \sim \frac{1}{\chi_n^{2k}\sqrt{N_{\text{evts}}}}. \quad (9)$$

with $\chi_n \equiv V_n\sqrt{M/\langle w^2\rangle}$. A thorough calculation, performed in Ref. [14], Appendix D, shows that this order of magnitude is indeed correct as long as χ_n is not larger than unity, which is the case for most experiments at ultrarelativistic energies.

In order to increase the statistics, it is possible to combine different runs performed in a given experiment, each of which has its own characteristics (different orientation of the magnetic field, etc.). For each of these runs, following the procedure described above leads to cumulants $c_{n,\alpha}\{2k\}$ (where α labels the run) and finally, accounting for the specific acceptance corrections, to flow estimates $V_{n,\alpha}\{2k\}^{2k}$. The proper way to combine the runs consists in averaging the $V_{n,\alpha}\{2k\}^{2k}$ (*not the* $V_{n,\alpha}\{2k\}$) of the various runs, weighted by the number of events in each run. Note that if statistical fluctuations are large, it may happen that a run yield a negative value of $V_{n,\alpha}\{2k\}^{2k}$. Such a run must nevertheless be included in the averaging procedure.

III. DIFFERENTIAL FLOW

Let us turn to differential flow, i.e., to the flow of an identified particle in a restricted portion of phase space (typically a narrow p_T or y interval). We call such a particle a "proton", and denote its azimuth by ψ; thus, the differential flow is $v'_n \equiv \langle e^{in(\psi-\Phi_R)}\rangle$, where the average value is taken over all protons.

Experimentally, the differential flow is obtained from the azimuthal correlations between the proton and the particles previously used to determine the integrated flow, which we call "pions". It is well known in the standard flow analysis that from the reaction plane reconstructed in harmonic n, one may reconstruct not only the corresponding v'_n, but also higher harmonics v'_{mn}, where m is an integer (in practice, $m = 1$ or 2). Here, this is done by correlating the proton with m pions, i.e., by measuring $\langle\exp[in(m\psi - \phi_1 - \cdots - \phi_m)]\rangle$. As in the case of integrated flow, nonflow contributions to this correlation can be eliminated by going to higher orders, i.e., by correlating the proton with $2k + m$ pions with $k \geq 0$ (in practice $k = 0$ or 1), and constructing a cumulant $d_{mn/n}\{2k + m + 1\}$. The subscript refers to the fact that v'_{mn} is measured by using pions in harmonic n, while the number in curly brackets is the order of the correlation: $2k + m$ pions and 1 proton.

This cumulant retains only the contributions from flow, proportional to $v'_{mn}V_n^{2k+m}$, and from $(2k+m+1)$-particle nonflow correlations, which scale like $\mathcal{O}(1/N^{2k+m})$: all lower order nonflow correlations have been removed. If the contribution of flow dominates, the cumulant $d_{mn/n}\{2k+m+1\}$ yields an estimate of v'_{mn}, which we naturally denote $v'_{mn/n}\{2k + m + 1\}$.

As in the case of integrated flow, we derive the cumulants $d_{mn/n}\{2k + m + 1\}$ using a generating function, namely

$$\mathcal{D}_{mn/n}(z) \equiv \frac{\langle e^{imn\psi}\,G_n(z)\rangle}{\langle G_n(z)\rangle}. \quad (10)$$

The cumulant $d_{mn/n}\{2k+m+1\}$ is the real part of the coefficient of $z^{*k}z^{k+m}/[k!(k+m)!]$ in the power-series expansion of $\mathcal{D}_{mn/n}(z)$. In the numerator of Eq. (10), the average is performed over all protons (i.e., an event with 2 protons is

counted twice; this was not stated correctly in [14]). On the other hand, the denominator is averaged over all events. If the proton is one of the "pions", i.e., if it was used in the calculation of the generating function Eq. (3), one should divide $G_n(z)$ by $1 + w_j(z^* e^{in\psi} + z e^{-in\psi})/M$, where ψ is the proton azimuth, to avoid autocorrelations. Note that while the number of pions in Eq. (3) was fixed, the number of protons must be allowed to fluctuate from event to event: to increase statistics, one should use all available protons.

To extract the cumulants, one computes the product $z^{*m}\mathcal{D}_{mn/n}(z)$ at the points $z_{p,q}$, Eq. (5); one then takes the real part, and averages over angles:

$$D_p \equiv \frac{(r_0\sqrt{p})^m}{q_{\max}} \sum_{q=0}^{q_{\max}-1} \left[\cos\left(m\frac{2\,q\,\pi}{q_{\max}}\right) X_{p,q} + \sin\left(m\frac{2\,q\,\pi}{q_{\max}}\right) Y_{p,q} \right], \tag{11}$$

with $p = 1, 2, 3$ and $X_{p,q} + iY_{p,q} \equiv \mathcal{D}_{mn/n}(z_{p,q})$. Note that although we present the integrated and differential flow analyses as two successive steps, they are in fact simultaneous: while the generating function $G_n(z)$ is calculated for a given event, one can at the same time compute its product by $e^{imn\psi}$ for the numerator of Eq. (10).

For $m = 1$ (useful for $v'_{1/1}$ or $v'_{2/2}$), the lowest order cumulants are given by

$$d_{n/n}\{2\} = \frac{1}{r_0^2}\left(2\,D_1 - \frac{1}{2}\,D_2\right), \qquad d_{n/n}\{4\} = \frac{1}{r_0^4}\left(-2\,D_1 + D_2\right), \tag{12}$$

while for $m = 2$, which can be used to derive $v'_{2/1}$,

$$d_{2n/n}\{3\} = \frac{1}{r_0^4}\left(4\,D_1 - \frac{1}{2}\,D_2\right), \qquad d_{2n/n}\{5\} = \frac{1}{r_0^6}\left(-6\,D_1 + \frac{3}{2}\,D_2\right). \tag{13}$$

These cumulants must then be related to the differential flow. For a perfect detector:

$$
\begin{aligned}
v'_{n/n}\{2\} &= d_{n/n}\{2\}/V_n, & v'_{n/n}\{4\} &= -d_{n/n}\{4\}/V_n^3, \\
v'_{2n/n}\{3\} &= d_{2n/n}\{3\}/V_n^2, & v'_{2n/n}\{5\} &= -d_{2n/n}\{5\}/(2V_n^4).
\end{aligned}
\tag{14}
$$

Note that these relations involve the integrated flow V_n obtained in the previous section. Relations for a nonisotropic acceptance are given in next Section. The order of magnitude of the statistical uncertainty can be estimated using the same arguments as for integrated flow. One obtains

$$\delta v'_{mn/n}\{2k + m + 1\} \sim \frac{1}{\chi_n^{2k+m}\sqrt{N'}}, \tag{15}$$

where N' is the number of protons used in the analysis. It may be worth noticing that the statistical error does not only depends on the available event statistics and multiplicity. It also strongly depends on the flow itself through the parameter χ_n.

IV. ACCEPTANCE CORRECTIONS

Our method can be used even when the detector used to measure the particle azimuths has only partial azimuthal coverage, provided that the event centralities be determined with an *independent* detector, as, e.g., a ZDC, with an approximately isotropic coverage. This is to make sure that the apparent multiplicity/centrality is not biased by the orientation of the reaction plane with respect to the detector.

A specific detector is characterized by its acceptance/efficiency function $A(j, \phi, p_T, y)$, which represents the probability that a particle of type j (pion, proton, etc.) with azimuth ϕ, transverse momentum p_T, and rapidity y, be detected. In practice, $A(j, \phi, p_T, y)$ is proportional to the number of hits in a (ϕ, p_T, y) bin, and thus can be obtained in a straightforward way while scanning through the data. The Fourier coefficients of the acceptance function are

$$A_n(j, p_T, y) \equiv \int_0^{2\pi} A(j, \phi, p_T, y)\, e^{-in\phi}\, \mathrm{d}\phi. \tag{16a}$$

These differential coefficients can be integrated, with appropriate weighting and a sum over the various types of particles used for the flow analysis, so as to describe the "integrated" acceptance of the detector:

$$a_n = \frac{\sum_j \int w(j, p_T, y) \, A_n(j, p_T, y) \, \mathrm{d}p_T \mathrm{d}y}{\sum_j \int w(j, p_T, y) \, A_0(j, pT, y) \, \mathrm{d}p_T \mathrm{d}y}. \tag{16b}$$

Note our introducing the weights $w(j, p_T, y)$, which are of course the same as in Eq. (3).

When the acceptance of a detector is not perfectly isotropic in ϕ, the cumulants Eq. (7) will mix different flow harmonics: $c_n\{2k\}$ does not depend only on V_n, but also on other V_p with $p \neq n$. For instance,

$$
\begin{aligned}
c_1\{4\} = &- \left[(1 - |a_1|^2)^4 + 4(1 - |a_1|^2)^2 |a_2 - a_1^2|^2 + |a_2 - a_1^2|^4\right] V_1^4 \\
&- \left[|a_1 - a_2 a_1^*|^4 + 4|a_1 - a_2 a_1^*|^2 |a_3 - a_1 a_2|^2 + |a_3 - a_1 a_2|^4\right] V_2^4,
\end{aligned} \tag{17a}
$$

and

$$
\begin{aligned}
c_2\{4\} = &- \left[|a_1 - a_2 a_1^*|^4 + 4|a_1 - a_2 a_1^*|^2 |a_3 - a_1 a_2|^2 + |a_3 - a_1 a_2|^4\right] V_1^4 \\
&- \left[(1 - |a_2|^2)^4 + 4(1 - |a_2|^2)^2 |a_4 - a_2^2|^2 + |a_4 - a_2^2|^4\right] V_2^4,
\end{aligned} \tag{17b}
$$

where we have assumed that all other flow harmonics $V_{n \geq 3}$ are negligible. The corresponding relations for the cumulants $c_1\{2\}$ and $c_2\{2\}$ can be found in Ref. [14], Appendix C1. If the detector acceptance is not too bad, the coefficients $a_{k \neq 0}$ will be small, and these expressions are close to Eqs. (8), valid for perfect detector.

Equations (17) form a linear system which can easily be inverted to express V_1^4 and V_2^4 (or, more precisely, the estimates $V_1\{4\}^4$ and $V_2\{4\}^4$) as functions of $c_1\{4\}$ and $c_2\{4\}$.

Let us now consider differential flow. Integrated and differential flows may be measured using two different detectors: for instance, a large acceptance detector for integrated flow, and a smaller one, but with better particle identification or p_T determination, for differential flow. For sake of generality, we thus denote by $A'(j, \psi, p_T, y)$ the corresponding acceptance function and by $A'_k(j, p_T, y)$ its Fourier coefficients defined as in (16a). The differential acceptance coefficients a'_k are then defined as in Eq. (16b), without the weights and the summation over j (since one usually measures the differential flow of identified particles) and with the integration over p_T and y restricted to the phase-space region under interest (typically, one integrates over p_T *or* y, so as to obtain v_n as a function of y or p_T, respectively).

Once again, anisotropies in the detector acceptance lead to some interference between the flow harmonics in the expressions of the cumulants $d_{mn/n}\{2k + m + 1\}$. Here are, for example, the relations between the lowest order cumulants and flow which allow one to extract the differential directed flow v'_1 and the differential elliptic flow v'_2 obtained either with respect to V_1 or V_2, i.e., what we denote $v'_{2/1}$ and $v'_{2/2}$ (relations for other cumulants can be found in Ref. [14], Appendix C2):

$$
\begin{aligned}
d_{1/1}\{4\} = &-\mathrm{Re}\left[(1 - |a_1|^2)\left((1 - |a_1|^2)^2 + 2|a_2 - a_1^2|^2\right) + (a'_2)^*(a_2 - a_1^2)\left(2(1 - |a_1|^2)^2 + |a_2 - a_1^2|^2\right)\right] v'_1 V_1^3 \\
&-\mathrm{Re}\left[a'_1(a_1^* - a_2^* a_1)\left(|a_1^* - a_2^* a_1|^2 + 2|a_3 - a_1 a_2|^2\right)\right. \\
&\left.+ (a'_3)^*(a_3 - a_1 a_2)\left(2|a_1^* - a_2^* a_1|^2 + |a_3 - a_1 a_2|^2\right)\right] v'_2 V_2^3,
\end{aligned} \tag{18a}
$$

$$
\begin{aligned}
d_{2/1}\{3\} = &\,\mathrm{Re}\left[(1 - |a_1|^2)^2 + (a'_4)^*(a_2 - a_1^2)^2\right] v'_2 V_1^2 \\
&+ 2\,\mathrm{Re}\left[(a'_1)^*(a_1^* - a_2^* a_1)(a_2 - a_1^2) + (a'_3)^*(a_3 - a_2 a_1)(1 - |a_1|^2)\right] v'_1 V_1 V_2,
\end{aligned} \tag{18b}
$$

$$
\begin{aligned}
d_{2/2}\{4\} = &-\mathrm{Re}\left[(1 - |a_2|^2)\left((1 - |a_2|^2)^2 + 2|a_4 - a_2^2|^2\right) + (a'_4)^*(a_4 - a_2^2)\left(2(1 - |a_2|^2)^2 + |a_4 - a_2^2|^2\right)\right] v'_2 V_2^3 \\
&-\mathrm{Re}\left[(a'_1)^*(a_1 - a_2 a_1^*)\left(|a_1^* - a_2^* a_1|^2 + 2|a_3 - a_1 a_2|^2\right)\right. \\
&\left.+ (a'_3)^*(a_3 - a_1 a_2)\left(2|a_1^* - a_2^* a_1|^2 + |a_3 - a_1 a_2|^2\right)\right] v'_1 V_1^3.
\end{aligned} \tag{18c}
$$

In these expressions, we have neglected terms involving higher flow harmonics $V_{n \geq 3}$ or $v'_{n \geq 3}$, and Re means that one should take the real part. Of course, they reduce to Eqs. (14) when the acceptance is isotropic.

Any two of Eqs. (18) constitute a linear system which can be inverted to obtain v'_1 and v'_2 once V_1 and V_2 have been extracted.

V. ADDITIONAL REMARKS AND COMMENTS

The cumulant method not only eliminates nonflow correlations, it also provides several independent estimates of the flow from cumulants of various orders. Since nonflow correlations between 4 or more particles are expected to be negligible, $V_n\{4\}$ and $V_n\{6\}$ should be consistent with each other within statistical error bars. With the high multiplicity produced at ultrarelativistic energies and the large acceptance detectors available, estimates show that one could even construct cumulants of 8-, 10-particle correlations. Since the generating function formalism yields all cumulant orders at once, one would simply need to increase the number of interpolation points $z_{p,q}$, which would result in a moderate increase in computer time. Checking that all higher order cumulants yield compatible values of V_n would give the first direct evidence that azimuthal correlations are of collective origin. Similarly, in the case of differential flow, various cumulants orders yield independent estimates which can be compared to one another.

Other consistency checks can be proposed to test the reliability of the results. First, one should perform the analysis with at least two values of the parameter r_0 entering Eq. (5), in order to check the stability against numerical errors. Then, one can try different values of M in Eq. (3), as, for instance, $M = 0.8 \langle M_{\text{tot}} \rangle$ and $M = 0.6 \langle M_{\text{tot}} \rangle$. That may also be a way to increase the statistics. In the same spirit, it is still possible to let M vary from event to event in Eq. (3), taking $M = M_{\text{tot}}$, especially if the acceptance is reasonably good. In that case, M must be replaced by the average $\langle M_{\text{tot}} \rangle$ in Eq. (4), and one should check that the results are consistent with what would be obtained with a fixed M.

The last consistency check regards differential flow. We have seen that the integrated flow V_n can be calculated using different weights w_j; the resulting weighted averages V_n's differ. However, the values of the differential flow $v'_{mn/n}\{2k+m+1\}$ obtained with different weights should be consistent within error bars if they are not contaminated by nonflow effects.

The recent STAR analysis [15] illustrates quite well the relevance of cumulants to the flow analysis: for integrated flow, the lowest order $V_n\{2\}$ reproduces the results of the standard two-particle methods, as it should. On the other hand, the higher order estimate $V_n\{4\}$ is lower than $V_n\{2\}$ beyond statistical error bars, especially for peripheral collisions, thereby suggesting that nonflow correlations are important. Although statistical uncertainties on higher order cumulants are larger, this loss is therefore compensated by the gain on errors from nonflow effects.

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