Anisotropic flow from Lee–Yang zeroes: a practical guide

N Borghini†, R S Bhalerao‡ and J-Y Ollitrault†

† Service de Physique Théorique, CEA-Saclay, F-91191 Gif-sur-Yvette cedex, France
‡ Department of Theoretical Physics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400 005, India

Abstract. We present a new method to analyze anisotropic flow from the genuine correlation among a large number of particles, focusing on the practical implementation of the method.

1. Introduction

Anisotropic collective flow is defined as a correlation between the azimuthal angle ϕ of an arbitrary particle and the azimuth Φ_R of the impact parameter in a non-central nucleus–nucleus collision. It is commonly characterized by the Fourier coefficients of the single-particle distribution [1]

$$v_n \equiv \langle \cos(n(\phi - \Phi_R)) \rangle, \tag{1}$$

where $\langle \ldots \rangle$ here denotes an average over many particles and events. In particular, elliptic flow v_2 is recognized as a sensitive probe of thermalization at RHIC [2].

While anisotropic flow is by definition a collective effect that involves many particles, it has mostly been analyzed using methods based either on a study of two-particle correlations [3–5] or on the cumulants of correlations between a few (in practice, up to 8) particles [6]. We recently proposed a new method of analysis [7, 8] that remedies this limitation, and extracts flow from the correlation between a large number of particles instead of only a few. In the following, we introduce the practical recipes for implementing the method, referring the reader to Refs. [7,8] for theoretical justifications.

2. Integrated flow

For a given centrality bin, the first step in the analysis is to obtain an estimate of the flow integrated over some phase-space region (typically corresponding to the acceptance of a detector or a set of detectors). We define *integrated flow* as the average over events

$$V_n \equiv \left\langle \sum_{j=1}^{M} w_j \cos(n(\phi_j - \Phi_R)) \right\rangle, \tag{2}$$

where the sum runs over all particles detected in an event. As is usual in flow analyses, w_j is a weight attributed to the j-th particle so as to maximize the integrated flow value. Further details regarding the choice of weights can be found in Ref. [6]. With unit weights, and neglecting for simplicity multiplicity fluctuations, $V_n = Mv_n$, where v_n is an average of the Fourier coefficient (1) over the phase space covered by the detector.

To derive an estimate of the integrated flow V_n (for practical purposes, $n = 2$ at RHIC and LHC, $n = 1$ at AGS and below), one should first compute for each event the complex-valued function‡

$$g^\theta(\mathrm{i}r) \equiv \prod_{j=1}^{M} [1 + \mathrm{i}r\, w_j \cos(n(\phi_j - \theta))] \tag{3}$$

for various values of the real positive variable r and of the angle θ ($0 \leq \theta < \pi/n$; in practice, 4 or 5 equally spaced values of θ are enough as shown in Ref. [8]). The ϕ_j are the *measured* azimuthal angles of the particles, using a fixed reference in the laboratory, and the product runs over *all* particles. Please note that the method is stable against effects, like multiple hits or showering, which bias the results of other methods of analysis, so that one should not refrain from using all detected particles, combining information from different detectors: increasing the multiplicity results in smaller statistical uncertainties on the flow estimates.

Together with values of $g^\theta(\mathrm{i}r)$, one should in each event compute the sums

$$Q_x \equiv \sum_{j=1}^{M} w_j \cos(n\phi_j), \qquad Q_y \equiv \sum_{j=1}^{M} w_j \sin(n\phi_j), \tag{4}$$

as well as their squares $Q_x{}^2$ and $Q_y{}^2$.

Next, one should average $g^\theta(\mathrm{i}r)$ over events for each value of r and θ:

$$G^\theta(\mathrm{i}r) \equiv \left\langle g^\theta(\mathrm{i}r) \right\rangle_{\mathrm{evts}} \equiv \frac{1}{N_{\mathrm{evts}}} \sum_{\mathrm{events}} g^\theta(\mathrm{i}r), \tag{5}$$

where N_{evts} is the number of events used in the analysis. This is also a good time to compute the averages over events $\langle Q_x \rangle$, $\langle Q_y \rangle$ and $\langle Q_x{}^2 + Q_y{}^2 \rangle$.

For every θ value, one must then look for the position r_0^θ of the first positive minimum of *the modulus* $|G^\theta(\mathrm{i}r)|$. An estimate of the integrated flow V_n is given by

$$V_n^\theta\{\infty\} \equiv \frac{j_{01}}{r_0^\theta}, \tag{6}$$

where $j_{01} \simeq 2.40483$ is the first zero of the Bessel function J_0. If the detector acceptance has reasonable azimuthal symmetry, the estimates do not depend on θ up to statistical fluctuations (see below).§ One eventually averages $V_n^\theta\{\infty\}$ over θ. This yields a new

‡ In Refs. [7,8] we used a different generating function. The one defined in Eq. (3), which is introduced in Appendix A of Ref. [8], actually yields more accurate results for higher harmonics v_{2n}, v_{3n} when analyzing differential flow. However, both forms of generating functions are actually equivalent for the most part, and the computations of Ref. [8] can easily be adapted to the present function Eq. (3).

§ For an anisotropic detector, $V_n^\theta\{\infty\}$ shows an oscillatory pattern that can be computed in terms of the Fourier coefficients of the detector acceptance-efficiency profile. There is an extra proportionality factor between $V_n\{\infty\}$ and V_n that can also be calculated and is close to unity in most cases, see Ref. [8] for details.

estimate $V_n\{\infty\}$ with smaller statistical errors. This estimate is then used to compute the resolution parameter, which measures the relative strength of flow compared to finite-multiplicity fluctuations: $\chi \equiv V_n\{\infty\}/\sigma$ [4], with σ given by

$$\sigma^2 \equiv \left\langle Q_x^{\,2} + Q_y^{\,2} \right\rangle - \langle Q_x \rangle^2 - \langle Q_y \rangle^2 - V_n\{\infty\}^2. \tag{7}$$

The relative statistical error on $V_n^\theta\{\infty\}$ due to the finite number of events is

$$\frac{\left\langle (\delta V_n^\theta\{\infty\})^2 \right\rangle}{V_n^2} = \frac{1}{2 N_{\mathrm{evts}}\, j_{01}^2\, J_1(j_{01})^2} \left[\exp\left(\frac{j_{01}^2}{2\chi^2}\right) + \exp\left(-\frac{j_{01}^2}{2\chi^2}\right) J_0(2 j_{01}) \right], \tag{8}$$

where J_1 is the spherical Bessel function of order 1. The statistical uncertainty on the estimate $V_n\{\infty\}$ is a about a factor of 2 smaller [8].

3. Differential flow

Once integrated flow estimates have been obtained, one can turn to the analysis of *differential flow*, i.e., the flow of particles of a given type in a definite phase-space window, which we shall call "protons" for the sake of brevity. A "proton" azimuth will be denoted by ψ, and the corresponding differential flow $v_p(p_T, y)$ by v_p'. Using an estimate of integrated flow in harmonic n, as e.g. $V_n^\theta\{\infty\}$, one can derive an estimate of v_p' in any harmonic p which is a multiple of n, i.e., $p = mn$ with m integer.

Now, for a given angle θ, with the help of the values of r_0^θ and $V_n^\theta\{\infty\}$ determined following the recipe of Sec. 2, an estimate of v'_{mn} is given by

$$\frac{v'^\theta_{mn}\{\infty\}}{V_n^\theta\{\infty\}} = \frac{J_1(j_{01})}{J_m(j_{01})} \, \mathrm{Re} \left(\frac{\left\langle g^\theta(\mathrm{i} r_0^\theta) \dfrac{\cos(mn(\psi - \theta))}{1 + \mathrm{i} r_0^\theta w_\psi \cos(n(\psi - \theta))} \right\rangle_\psi}{\mathrm{i}^{m-1} \left\langle g^\theta(\mathrm{i} r_0^\theta) \displaystyle\sum_j \dfrac{w_j \cos(n(\phi_j - \theta))}{1 + \mathrm{i} r_0^\theta w_j \cos(n(\phi_j - \theta))} \right\rangle_{\mathrm{evts}}} \right). \tag{9}$$

In the denominator, the average is over events and the sum runs over all particles in each event. By contrast, the average $\langle \ldots \rangle_\psi$ in the numerator is over *protons*, and we have denoted by w_ψ the weight associated with a proton. Please note that the sum in the denominator need only be computed once per event (it is actually the derivative of $g^\theta(\mathrm{i} r)$ at the minimum r_0^θ), while the quantity to be averaged in the numerator varies from one proton to the other, even for protons within the same event. Finally, Re denotes the real part of the (complex-valued) ratio.

Denoting by N' the total number of "protons" in the phase-space bin under study, the statistical uncertainty on the estimate $v'^\theta_{mn}\{\infty\}$ is

$$\left\langle (\delta v'^\theta_{mn}\{\infty\})^2 \right\rangle = \frac{1}{4 N' J_m(j_{01})^2} \left[\exp\left(\frac{j_{01}^2}{2\chi^2}\right) + (-1)^m \exp\left(-\frac{j_{01}^2}{2\chi^2}\right) J_0(2 j_{01}) \right]. \tag{10}$$

As in the case of integrated flow, averaging the various estimates $v'^\theta_{mn}\{\infty\}$ results in a new estimate $v'_{mn}\{\infty\}$ with reduced statistical error bars (by a factor $\simeq 2$). Regarding the effects of detector anisotropies, they are the same as above too [8]: a θ-dependence of $v'^\theta_{mn}\{\infty\}$ and a multiplicative factor between the "true" v'_{mn} and its estimate $v'_{mn}\{\infty\}$.

Finally, let us briefly mention systematic errors inherent to the method. A careful study in Ref. [8] allowed us to conclude that the relative error due the interplay of nonflow effects and higher harmonics (in particular v_{2n}) is of order

$$\frac{\delta v'_{mn}\{\infty\}}{v'_{mn}} = O\left(\frac{1}{M}\right) + O\left(\frac{1}{(Mv_n)^2}\right) + O\left(\frac{v_{2n}}{Mv_n^2}\right). \tag{11}$$

The relative error on integrated flow estimate $V_n\{\infty\}$ is of the same order of magnitude. Actually, in the analysis of higher harmonics ($m > 1$), an extra error term arises

$$\delta v'_{mn}\{\infty\} = O\left(\frac{v'_{(m-1)n}}{Mv_n}\right). \tag{12}$$

Unlike the previous term, this is an absolute systematic error, not a relative error on the flow. With any other method of flow analysis, the systematic error will always be larger (or at least equal).

4. Discussion

The method of analysis we presented above is simple to implement: compute values of the generating function (5) and find the first minimum of its modulus, then Eq. (6) gives you the integrated flow. Knowing the position of the minimum, you can then compute in a second pass through the data the quantities in the right-hand side of Eq. (9) and deduce differential flow. Nothing more is required.

In addition, the method is conceptually rich. The minimum of $|G^\theta(ir)|$ is in fact compatible with a zero of $G^\theta(z)$, where z is a complex variable. The mere existence of such a zero *close to the origin and scaling with the inverse of the system size* (V_n in Eq. (2) is roughly proportional to the multiplicity M) signals the presence of collective effects. This is analogous to Lee–Yang theory of phase transitions [9] in which the zeroes of the partition function come closer to the origin with increasing system size if, and only if, there is a phase transition. Since anisotropic flow is a collective effect, Lee–Yang zeroes definitely are the most natural method to analyze flow.

References

[1] Voloshin S and Zhang Y 1996 *Z. Phys.* C **70** 665
[2] Gyulassy M 2004 these proceedings
[3] Danielewicz P and Odyniec G 1985 *Phys. Lett.* B **157** 146
[4] Ollitrault J-Y 1997 *Preprint* nucl-ex/9711003
[5] Poskanzer A M and Voloshin S A 1998 *Phys. Rev.* C **58** 1671
[6] Borghini N, Dinh P M and Ollitrault J-Y 2001 *Phys. Rev.* C **64** 054901
[7] Bhalerao R S, Borghini N and Ollitrault J-Y 2004 *Phys. Lett.* B **580** 157
[8] Bhalerao R S, Borghini N and Ollitrault J-Y 2003 *Nucl. Phys.* A **727** 373
[9] Yang C N and Lee T D 1952 *Phys. Rev.* **87** 404