Robust Covariance Matrix Estimation

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A. Estimating the Covariance Matrix

We have an M-element array with elements located along the x-axis at positions $d_x m$ for $m = 0, 1, \cdots, M-1$. They sample a wavefield $s(x)$ consisting of P sources arriving from angles θ_p. The signal observed by the m^{th} array element is:

$$a_m = s(x_m) + n_m = \sum_{p=0}^{P-1} A_p e^{ik(\theta_p)d_x m} + n_m, \quad (1)$$

for $m = 0, 1, \cdots, M-1$. The P sources contribute to the signal in the form of complex exponentials with spatial frequencies $\tilde{k}(\theta_p) = \frac{2\pi}{\lambda}\sin\theta_p d_x$. The sequence n_m represents spatially white noise (element self-noise), which is defined through:

$$E\left\{n_m\right\} = 0, E\left\{n_m n_{m+l}^*\right\} = \sigma_n^2 \delta_l, \quad (2)$$

where δ_l is the Kronecker delta function for index l.

Our goal is to use the array data vector, $\vec{a} = \left[a_0, \cdots, a_{M-1}\right]^T$, to estimate one or more of the unknown parameters of the wavefield, e.g. the complex amplitudes $\left\{A_p\right\}_{p=0}^{P-1}$ or the directions of arrival $\left\{\theta_p\right\}_{p=0}^{P-1}$ using adaptive array processing. In most adaptive array processing methods, it is necessary to estimate the covariance matrix of the array:

$$\mathbf{R} = E\left\{\vec{a}\vec{a}^H\right\}. \quad (3)$$

An element in the covariance matrix is given by:

$$\begin{aligned}\left[\mathbf{R}\right]_{m,n} &= E\left\{a_m a_n^*\right\} = E\left\{s(x_m)s^*(x_n)\right\} \\ &= E\left\{s(x_m)s^*(x_m + \Delta_x[m,n])\right\}, \end{aligned} \quad (4)$$

where $\left[\mathbf{R}\right]_{m,n}$ means the element in row m and column n of the matrix $\mathbf{R}$, and $\Delta_x[m,n]$ is the spatial lag (i.e. distance) between array elements m and n:

$$\Delta_x[m,n] = x_m - x_n. \quad (5)$$

For ULAs, the lag function is very simple:

$$\Delta_x[m,n] = d_x l \text{ for } l = m - n. \quad (6)$$

A wavefield of monochromatic plane waves is *stationary*, meaning that the spatial correlation depends on lag only:

$$E\left\{s(x_m)s^*(x_m + d_x l)\right\} = E\left\{s(x)s^*(x + d_x l)\right\} \forall x \forall l \quad (7)$$

Therefore, we can define the spatial correlation function of a ULA with the single parameter $l = m - n$:

$$\left[R\right]_{m,n} = r[l] = E\left\{s(0)s^*(d_x l)\right\} \quad (8)$$

Since $s(x)$ can be written as a function of its DTFT $S(\tilde{k})$, we can rewrite the spatial correlation function as:

$$r[l] = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} E\left\{S(\tilde{k})S^*(\tilde{k}')\right\} e^{-i\tilde{k}'l}\, d\tilde{k}'\, d\tilde{k}. \quad (9)$$

Because $S(\tilde{k})$ is bandlimited as specified in (**??**), we can rewrite the integration limits:

$$r[l] = \frac{1}{4\pi^2}\int_{-\kappa\pi}^{\kappa\pi}\int_{-\kappa\pi}^{\kappa\pi} E\left\{S(\tilde{k})S^*(\tilde{k}')\right\} e^{-i\tilde{k}'l}\, d\tilde{k}'\, d\tilde{k}. \quad (10)$$

The expectation in the above expression depends on the actual statistics of the complex plane wave amplitudes A_p. The basis for most algorithms such as MVDR, MUSIC, ESPRIT, etc. is the assumption that they are uncorrelated:

$$E\left\{A_p A_{p'}^*\right\} = \left|A_p\right|^2 \delta_{p-p'}. \quad (11)$$

This implies that the wavefield is uncorrelated across wavenumbers:

$$E\left\{S(\tilde{k})S^*(\tilde{k}')\right\} = \left|S(\tilde{k})\right|^2 \delta(\tilde{k} - \tilde{k}') \quad (12)$$

In reality, this may not be the case. However, when they are uncorrelated the inverse DTFT in (10) reduces to:

$$r[l] = \frac{1}{2\pi}\int_{-\kappa\pi}^{\kappa\pi} \left|S(\tilde{k})\right|^2 e^{-i\tilde{k}l}\, d\tilde{k}. \quad (13)$$

We will use this expression for our covariance matrix estimator. Since we do not know $S(\tilde{k})$, we will first need to estimate it. An obvious estimator is DAS, as given in (**??**) with $\tilde{w} = M^{-1}$. Inserting (**??**) into (13) and carrying out the integration yields:

$$r[l] = M^{-2}\sum_{m=0}^{M-1}\sum_{n=0}^{M-1} a_m a_n \frac{\sin(\kappa\pi(m-n))}{m-n} \quad (14)$$

For the special case of $d_x = \frac{\lambda}{2}$, this reduces to:

$$r[l] = M^{-1}\sum_{m=0}^{M-l-1} a_m a_{m+l}, \quad (15)$$

which is the sum along the l^{th} diagonal of the sample covariance matrix, divided by the number of elements in the array.

B. Additional points

1) Spatially white noise is not covered by the current method. Introduce via diagonal loading?
2) Importance of including κ in integration limits, cf. (10).

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