Improved Forward-Backward Averaging for the APES Beamformer

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***Abstract*—The Minimum Variance Distortionless Response (MVDR) beamformer has good performance, but it requires knowledge of the spatial noise-and-interference covariance matrix. This is generally not available, but can be estimated from data using signal subtraction as in Spatial APES (S-APES). Forward-backward (FB) averaging is often applied to improve the performance of S-APES. In this letter we suggest an improved technique termed full forward-backward averaging (FFB). We prove that FFB is superior to FB in the sense that it increases the rank of the covariance matrix estimate by up to 50%, and also improves the convergence of the noise covariance matrix estimate. We demonstrate that FFB can be used to attain lower RMSE than FB in S-APES.**

I. Introduction

Adaptive beamformers based on the criterion of Minimum Variance-Distortionless Response (MVDR) are ubiquitous in spatial signal processing. Unfortunately, they depend on knowledge of the interference-plus-noise covariance matrix, which is rarely available in practice. One possible way of estimating the noise statistics is through signal subtraction; the signal of interest is first estimated using conventional means and subsequently subtracted from the data vectors, thereby forming noise vector estimates. A well-known algorithm utilizing this approach is the Amplitude and Phase EStimation (APES) technique. APES was introduced as an alternative to the Capon spectral estimator based on an approximate maximum likelihood criterion [1], and has later been reinterpreted as a deterministic optimal estimation method [2] and as a matched filterbank [3]. APES can be combined with various techniques to improve its robustness [4], [5]. The extension of APES to beamforming is referred to as multilook APES [6] or Spatial APES (S-APES) [7]. Repeated investigations of S-APES have shown that it generally exhibits superior estimation accuracy and robustness compared to the Capon beamformer, at the cost of decreased resolution. Both Capon and S-APES can make use of a technique known as forward-backward (FB) averaging [8], [9] to improve covariance matrix estimates.

In this letter we suggest the new full forward-backward (FFB) averaging scheme for S-APES, and prove that it can increase the rank of the covariance matrix estimate by up to 50% as compared to conventional FB. We also prove that the covariance matrix estimate converges faster using FFB than FB. Finally, we demonstrate through simulations that FFB can be used with S-APES to achieve lower RMSE than when conventional FB is used.

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II. Background

A. Signal Model and MVDR Beamforming

We investigate a narrowband signal impinging on an M-element uniform linear array (ULA) in N temporal snapshots:

$$\vec{x}[n] = \vec{s}[n] + \vec{q}[n] \in \mathbb{C}^M, n = 0, 1, \cdots, N-1 \quad (1)$$

where:

$$\vec{s}[n] = A_s\vec{v}_s[n], \vec{q}[n] = \vec{e}[n] + \vec{n}[n], \vec{e}[n] = \sum_{d=1}^{D} A_d\vec{v}_d[n]. \quad (2)$$

The parameter θ depends on the source's direction of arrival, wavenumber, and the inter-element spacing of the ULA. The signal of interest is $\vec{s}$, and $\vec{e}$ is the sum of interfering signals. The vectors:

$$\vec{v}[n] = e^{i\omega n}\left[e^{-i\theta(M-1)/2}, \cdots, 1, \cdots, e^{i\theta(M-1)/2}\right]^T \quad (3)$$

are the propagation vectors associated with the different signals. A_s and A_d are the (possibly correlated) complex signal amplitudes. The noise $\vec{n}$ is assumed spatially white with circular symmetric complex normal entries. The data, interference-plus-noise, interference, and noise covariance matrices, respectively, are:

$$\begin{aligned} &\mathbf{R} = E\left\{\vec{x}\vec{x}^H\right\}, \mathbf{Q} = E\left\{\vec{q}\vec{q}^H\right\}, \mathbf{E} = E\left\{\vec{e}\vec{e}^H\right\} \\ &\mathbf{N} = E\left\{\vec{n}\vec{n}^H\right\} = \sigma_n^2\mathbf{I} \end{aligned} \quad (4)$$

Our goal is to estimate A_s from $\vec{x}[n]$, under the assumption that the signal's direction of arrival, and therefore $\vec{v}_s$, is known. This is accomplished through beamforming:

$$\hat{A}_s = \vec{w}^H\vec{x} \quad (5)$$

The weight vector $\vec{w}$ can be chosen statically, such as the uniformly weighted delay-and-sum (DAS) beamformer:

$$\vec{w}_{DAS} = \frac{1}{M}\vec{v}_s, \quad (6)$$

but better estimates may be achieved with adaptive beamformers. In this paper we concentrate on the optimal MVDR beamformer, which has the following weights:

$$\vec{w}_{MVDR} = \left(\arg\min_{\vec{w}} \vec{w}^H\mathbf{Q}\vec{w} \text{ s.t. } \vec{w}^H\vec{v}_s = 1\right) = \frac{\mathbf{Q}^{-1}\vec{v}_s}{\vec{v}_s^H\mathbf{Q}^{-1}\vec{v}_s}. \quad (7)$$

For low-SNR scenarios, $\vec{w}_{MVDR}$ converges to the uniform weight vector of (6). In practice, $\mathbf{Q}$ is usually unknown and must be estimated. This is the aim of the APES method, which is covered in the following section.

B. APES Beamformer

The APES spectral estimator was introduced by Li and Stoica in [1] as a maximum-likelihood amplitude estimator. A multi-look extension, suited for beamforming, was introduced by Gini and Lombardini [6] as Multilook-APES (MAPES). We will follow Jakobsson and Stoica [7] and refer to this method as Spatial APES (S-APES). S-APES can be interpreted as an implementation of (7) [3], in which the interference-plus-noise covariance matrix $\mathbf{Q}$ is estimated by first dividing the array into K overlapping subarrays of length L:

$$\vec{x}_k[n] = \left[x_k[n], \cdots, x_{k+L-1}[n]\right]^T, \tag{8}$$

and then averaging the subarray covariance matrixes over time and space, while subtracting a signal estimate:

$$\begin{aligned}\hat{\mathbf{Q}} &= \frac{1}{KN}\sum_{n=0}^{N-1}\sum_{k=0}^{K-1}\left(\vec{x}_k[n]e^{-i\theta k} - \vec{m}[n]\right)\left(\vec{x}_k[n]e^{-i\theta k} - \vec{m}[n]\right)^H \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\left(\hat{\mathbf{R}}[n] - \vec{m}[n]\vec{m}[n]^H\right) = \frac{1}{N}\sum_{n=0}^{N-1}\hat{\mathbf{Q}}[n]\end{aligned} \tag{9}$$

The signal estimate vector is found through the uniformly weighted delay-and-sum beamformer:

$$\vec{m}[n] = \frac{1}{K}\sum_{k=0}^{K-1}\vec{x}_k[n]e^{i\theta k}, \tag{10}$$

and the sample covariance matrix is given by:

$$\hat{\mathbf{R}}[n] = \frac{1}{K}\sum_{k=0}^{K-1}\vec{x}_k[n]\vec{x}_k[n]^H. \tag{11}$$

The S-APES estimate of A_s is given by inserting (9) into (7), and applying the resulting weights across all subarrays:

$$\hat{A}_s = \frac{\vec{d}_s^H\hat{\mathbf{Q}}^{-1}\vec{m}_f}{\vec{d}_s^H\hat{\mathbf{Q}}^{-1}\vec{d}_s} \tag{12}$$

For notational simplicity, the S-APES covariance matrix estimate can be rewritten as:

$$\hat{\mathbf{Q}} = \frac{1}{KN}\sum_{n=0}^{N-1}\sum_{k=0}^{K-1}\mathbf{B}_k^H\vec{x}[n]\vec{x}[n]^H\mathbf{B}_k, \tag{13}$$

where

$$\mathbf{B}_k = \mathbf{A}_k - \mathbf{C}, \tag{14}$$

$$\mathbf{A}_k = \left[\vec{\delta}_k e^{-i\theta k}, \cdots, \vec{\delta}_{k+L-1}e^{-i\theta k}\right]$$

$$\mathbf{C} = \frac{1}{K}\left[\sum_{k=0}^{K-1}\vec{\delta}_k e^{-i\theta k}, \cdots, \sum_{k=0}^{K-1}\vec{\delta}_{k+L-1}e^{-i\theta k}\right] \tag{15}$$

The vectors $\vec{\delta}_k$ have a 1 as their k^{th} element and zeros everywhere else. It can be shown that S-APES converges to MVDR as the number of subarrays, K, approaches infinity while the number of temporal samples, N, stays finite. Therefore, we now consider the more interesting and less researched case where $N \to \infty$ and K is finite. For $N = 1$, it is often assumed that $(L = M/2, K = M/2)$ yields the best performance [1]. Gini and Lombardini [6] have shown that when multiple temporal snapshots are available, $(L = M - 1, K = 2)$ is preferable. The total number of data vectors must at least satisfy $KN \geq L$ for $\hat{\mathbf{Q}}$ to be invertible. Arrays with persymmetric covariance matrices, such as ULAs, can utilize *forward-backward averaging* to double the number of outer products to average in (11), thereby improving covariance matrix estimation [8], [9], [10]. We say that a covariance matrix is persymmetric if it satisfies:

$$\mathbf{R} = \mathbf{J}\mathbf{R}^*\mathbf{J} \Leftrightarrow \mathbf{J}\vec{e}^* = \vec{e} \text{ and } \mathbf{J}\mathbf{N}^*\mathbf{J} = \mathbf{N}, \tag{16}$$

where $\mathbf{J}$ is the exchange matrix with elements equal to 1 on its antidiagonal and all other elements equal to 0. The $\hat{\mathbf{R}}$ in (11) is referred to as the forward-only covariance matrix. The backward-only covariance matrix is equal to:

$$\hat{\mathbf{R}}_b = \frac{1}{KN}\sum_{n=0}^{N-1}\sum_{k=0}^{K-1}\mathbf{B}_k^H\mathbf{J}\vec{x}^*[n]\vec{x}^T[n]\mathbf{J}\mathbf{B}_k \tag{17}$$

where the vectors $\mathbf{J}\vec{x}^*[n]$ are referred to as the backward data vectors. The forward-backward covariance matrix is defined as the average of the forward and backward covariance matrices:

$$\hat{\mathbf{R}}_{fb} = \frac{1}{2}\left(\hat{\mathbf{R}}_f + \hat{\mathbf{R}}_b\right). \tag{18}$$

Similarly, the forward-backward interference-plus-noise covariance matrix has been defined as the average of the forward and backward matrices [1], [2], [7], [11]:

$$\begin{aligned}\hat{\mathbf{Q}}_{fb}[n] &= \frac{1}{2}\left(\hat{\mathbf{Q}}_f[n] + \hat{\mathbf{Q}}_b[n]\right) \\ &= \hat{\mathbf{R}}_{fb}[n] - \frac{1}{2}\left(\vec{m}_f[n]\vec{m}_f[n]^H + \vec{m}_b[n]\vec{m}_b[n]^H\right),\end{aligned} \tag{19}$$

where $\hat{\mathbf{Q}}_f[n]$ and $\vec{m}_f[n]$ are the covariance matrix and signal vector estimators given by (9) and (10) , and $\hat{\mathbf{Q}}_b[n]$ and $\vec{m}_b[n]$ are calculated in the same way from the backward data vectors.

III. Improved Forward-Backward Averaging for S-APES

Unlike for the case of the forward-backward data covariance matrix of (18), there are actually several plausible ways of combining the forward and backward data vectors when estimating the interference-plus-noise covariance matrix, and (19) is not necessarily the optimal one. We suggest the following new estimator, which we call the interference-plus-noise covariance matrix with *full forward-backward averaging*:

$$\hat{\mathbf{Q}}_{ffb}[n] = \hat{\mathbf{R}}_{fb}[n] - \vec{m}_{fb}[n]\vec{m}_{fb}^H[n] \tag{20}$$

$$\text{for } \vec{m}_{fb}[n] = \frac{1}{2}\left(\vec{m}_f[n] + \vec{m}_b[n]\right).$$

Because the utilization of forward-backward data vectors is inherently ad-hoc in the sense that it is not derived from any optimization criterion, one cannot claim that either (19) or (20) is the more correct definition. However, we present the following propositions that demonstrate the superiority of FFB over FB in two important aspects.

Proposition 1: Assume the array data vectors are i.i.d. circular symmetric Gaussian and the forward- and backward subarray data vectors are distributed so that any subset of L

vectors or less are linearly independent. Then, the ranks of the FB and FFB estimators satisfy the following relation:

$$\text{rank}\left\{\hat{\mathbf{Q}}_{ffb}\right\} \geq \text{rank}\left\{\hat{\mathbf{Q}}_{fb}\right\} \tag{21}$$

Proof: From (19) we see that $\hat{\mathbf{Q}}_{fb}$ is the average of two matrices with individual ranks $\min\{L, KN-1\}$ because of the mean subtraction. Due to the independence assumption, the sum has rank $\min\{L, 2(KN-1)\}$. From (20) we see that $\hat{\mathbf{Q}}_{ffb}$ can be written as an average over $2KN$ samples minus their mean:

$$\hat{\mathbf{Q}}_{ffb} = \frac{1}{2KN}\sum_{n=0}^{N-1}\sum_{k=0}^{K-1}\left(\vec{x}_k[n] - \vec{m}_{fb}[n]\right)\left(\vec{x}_k[n] - \vec{m}_{fb}[n]\right)^H + \frac{1}{2KN}\sum_{n=0}^{N-1}\sum_{k=0}^{K-1}\left(\mathbf{J}\vec{x}_k^*[n] - \vec{m}_{fb}[n]\right)\left(\mathbf{J}\vec{x}_k^*[n] - \vec{m}_{fb}[n]\right)^H \tag{22}$$

Therefore, it has rank $\min\{L, 2KN-1\}$ due to the mean subtraction. Hence, we have:

$$\text{rank}\left\{\hat{\mathbf{Q}}_{ffb}\right\} = \min\{L, 2NK-1\} \geq \min\{L, 2N(K-1)\} = \text{rank}\left\{\hat{\mathbf{Q}}_{fb}\right\} \tag{23}$$

In particular, for $L = M-1$ and $3N \leq L$, the rank increases by 50% from $2N$ for FB to $3N$ for FFB. ■

A corollary of Proposition 1 is that FFB requires fewer temporal snapshots than FB to attain an invertible sample covariance matrix. In the case of single snapshot beamforming (or simply temporal APES), the implication is that FFB supports a larger subarray size than FB.

Proposition 2: The FB/FFB interference and noise covariance matrix estimators have the following properties:

$$\hat{\mathbf{E}}_{ffb} = \hat{\mathbf{E}}_{fb}$$
$$\left\|E_N\left\{\hat{\mathbf{N}}_{ffb}\right\} - \sigma_n^2\mathbf{I}\right\|_F \leq \left\|E_N\left\{\hat{\mathbf{N}}_{fb}\right\} - \sigma_n^2\mathbf{I}\right\|_F, \tag{24}$$

where $E_N\{\cdot\}$ denotes temporal expectation. In other words, the FB and FFB estimators achieve identical interference covariance matrices, while the noise covariance matrix estimate of FFB is closer to its true value than FB in the sense of the Frobenius norm. Therefore, $\hat{\mathbf{Q}}_{ffb}$ converges faster to $\mathbf{Q}$ than $\hat{\mathbf{Q}}_{fb}$ as K increases.

Proof: The FB interference covariance matrix is:

$$\hat{\mathbf{E}}_{fb} = \frac{1}{2NK}\sum_{n=0}^{N-1}\sum_{k=0}^{K-1}\mathbf{B}_k^H\left(\vec{e}[n]\vec{e}[n]^H + \mathbf{J}\vec{e}^*\vec{e}^T\mathbf{J}\right)\mathbf{B}_k = \frac{1}{NK}\sum_{n=0}^{N-1}\sum_{k=0}^{K-1}\mathbf{B}_k^H\vec{e}[n]\vec{e}[n]^H\mathbf{B}_k. \tag{25}$$

The FFB interference covariance matrix is:

$$\hat{\mathbf{E}}_{fb} = \frac{1}{2NK}\sum_{n,k}\left[S_k(\vec{e}[n])S_k^H(\vec{e}[n]) + S_k(\vec{e}^*[n])S_k^H(\vec{e}^*[n])\right] = \frac{1}{NK}\sum_{n=0}^{N-1}\sum_{k=0}^{K-1}\mathbf{B}_k^H\vec{e}[n]\vec{e}[n]^H\mathbf{B}_k \tag{26}$$

where $S_k(\vec{e}[n]) = \mathbf{A}_k^H\vec{e} - \frac{1}{2}\mathbf{C}^H\left(\vec{e} + \mathbf{J}\vec{e}^*\right)$, ergo the two matrices are equal. The results (25) and (26) follow directly from the fact that $\mathbf{J}\vec{e}^* = \vec{e}$.

The temporal expectation of the FB noise covariance matrix reduces to:

$$E_N\left\{\hat{\mathbf{N}}_{fb}\right\} = \frac{1}{2K}\sum_{k=0}^{K-1}\mathbf{B}_k^H E_N\left\{\vec{n}\vec{n}^H + \mathbf{J}\vec{n}^*\vec{n}^T\mathbf{J}\right\}\mathbf{B}_k = \frac{\sigma_n^2}{K}\sum_{k=0}^{K-1}\mathbf{B}_k^H\mathbf{B}_k \tag{27}$$

This is a Toeplitz matrix with elements:

$$\left[E_N\left\{\hat{\mathbf{N}}_{fb}\right\}\right]_{m,n} = \sigma_n^2\left(\delta_l + \gamma(l) - \frac{1}{K}\sum_{k=0}^{K-1}\beta(k,l)\right),$$
$$\text{where } \gamma(l) = \max\left\{0, \frac{K-l}{K^2}\right\}$$
$$\text{and } \beta(k,l) = \begin{cases} \frac{1+\delta_{k-l}}{K} & |k-l| < L \\ 0 & |k-l| \geq L \end{cases} \text{ and } l \triangleq |m-n|. \tag{28}$$

This yields the Frobenius-norm:

$$\left\|E_N\left\{\hat{\mathbf{N}}_{fb}\right\} - \sigma_n^2\mathbf{I}\right\|_F = \left(\sum_{m,n}\left|\gamma(l) - \frac{1}{K}\sum_{k=0}^{K-1}\beta(k,l)\right|^2\right)^{\frac{1}{2}}, \tag{29}$$

where l is defined as in (28). The temporal expectation of the FFB noise covariance matrix reduces to:

$$E_N\left\{\hat{\mathbf{N}}_{fb}\right\} = \frac{1}{2K}\sum_{k=0}^{K-1}E_N\left\{S_k(\vec{n})S_k(\vec{n})^H + S_k(\vec{n}^*)S_k(\vec{n}^*)^H\right\} = \frac{\sigma_n^2}{K}\sum_{k=0}^{K-1}\left[\left(\mathbf{A}_k - \frac{1}{2}\mathbf{C}\right)^H\left(\mathbf{A}_k - \frac{1}{2}\mathbf{C}\right) + \frac{1}{4}\mathbf{C}^H\mathbf{C}\right], \tag{30}$$

which is the Toeplitz matrix:

$$\left[E_N\left\{\hat{\mathbf{N}}_{ffb}\right\}\right]_{m,n} = \sigma_n^2\left(\delta_l + \frac{1}{2}\gamma(l) - \frac{1}{2K}\sum_{k=0}^{K-1}\beta(k,l)\right), \tag{31}$$

Inserting these expressions into (29), we find:

$$\left\|E_N\left\{\hat{\mathbf{N}}_{ffb}\right\} - \sigma_n^2\mathbf{I}\right\|_F = \frac{1}{2}\left\|E_N\left\{\hat{\mathbf{N}}_{fb}\right\} - \sigma_n^2\mathbf{I}\right\|_F \tag{32}$$

Asymptotically, for $K \to \infty$, we get:

$$\lim_{K\to\infty} E_N\left\{\hat{\mathbf{N}}_{ffb}\right\} = E_N\left\{\hat{\mathbf{N}}_{fb}\right\} = \sigma_n^2\mathbf{I} \tag{33}$$

In other words, both estimators converge spatially to the true noise covariance matrix. Proposition 2 follows directly from (32) and (33). ■

A corollary of Proposition 2 is that, as SNR decreases, neither FB-APES nor FFB-APES will converge (temporally) to the uniform weight vector (unlike the true MVDR beamformer). However, FFB-APES will yield a closer match than FB-APES. This is verified in Fig. 1 for $M = 10, L = M-1, \frac{|A_s|^2}{\sigma_n^2} \to 0$.

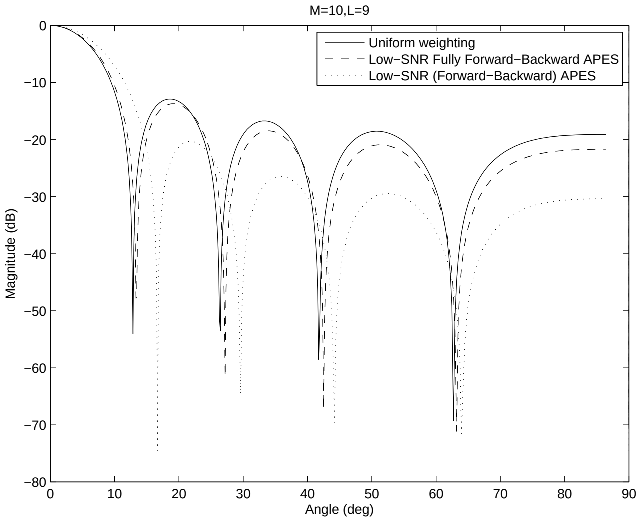

Fig. 1. Low-SNR response for true MVDR (uniform weighting), APES with forward-backward averaging, and APES with full forward-backward averaging.

Having show that $\hat{\mathbf{Q}}_{ffb}$ is superior to $\hat{\mathbf{Q}}_{fb}$ with respect to rank and white noise estimation, we propose that basing S-APES on FFB will improve performance. We test our hypothesis for the simple case of two unit-amplitude signals with variable angular spacing. The results are shown in Fig. 2. We see that FB leads to a decreased Root Mean Squared Error (RMSE) of A_s compared to forward-only, and FFB leads to a decreased RMSE compared to FB. In Fig. 2 **a)**, we see that FB performs better than forward-only, while FFB performs better than FB. In Fig. 2 **b)**, we see that, as the number of subarrays K increases, the RMSE values of the different beamformers converge.

IV. Conclusions

We have suggested a new method for forward-backward averaging in S-APES, termed full forward-backward averaging. The new method is proved to be superior to the conventional method with respect to rank and convergence. It is verified through simulations that full forward-backward averaging can result in lower RMSE than conventional forward-backward averaging.

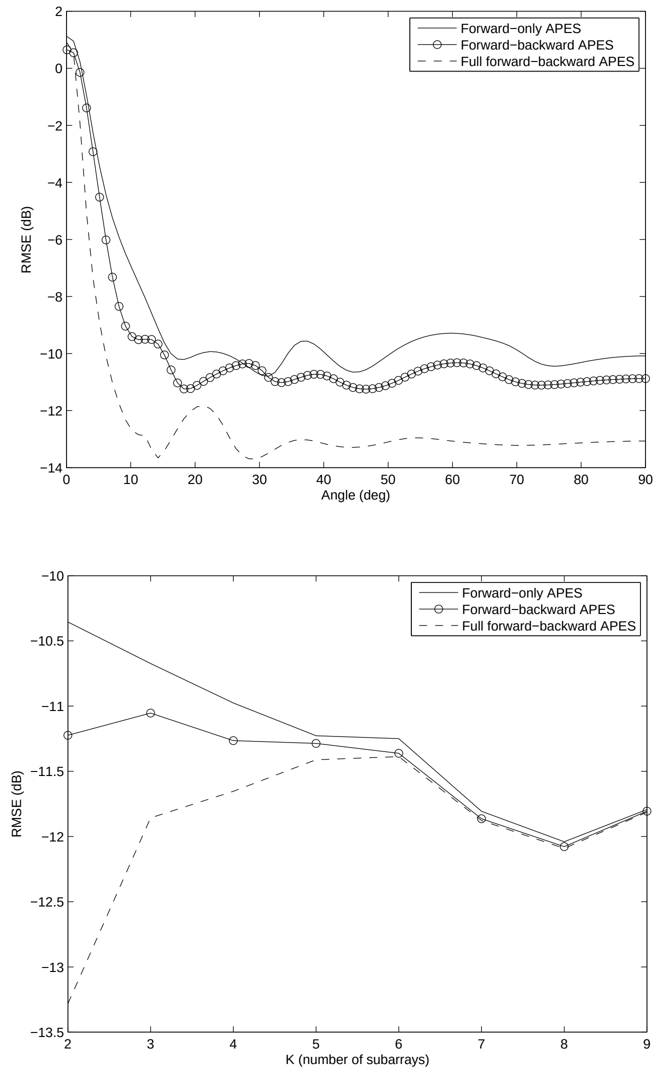

Fig. 2. Effect of different forward-backward averaging schemes on S-APES RMSE. Signal of interest arriving from 0° with $A_s = 1$. Interfering signal arriving from various angles with identical amplitude. N=10, SNR=0dB. **a)** RMSE vs. interference direction of arrival. **b)** RMSE vs. number of subarrays for interference from 45°.

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