Flow analysis with Q-cumulants

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Anisotropic flow measurements in heavy-ion collisions provide important constraints on the equation of state of hot and dense matter. In these measurements nonflow correlations need to be eliminated and therefore advanced multi-particle correlation techniques have been developed. More recently the importance of event by event fluctuations in the magnitude of the anisotropic flow has become clear. The anisotropic flow analysis using cumulants has a characteristic sensitivity to these fluctuations. Unfortunately the currently available approaches to calculate the cumulants have biases which complicate the analysis. In this paper we present a new *analytic* unbiased way to calculate the cumulants.

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I. INTRODUCTION

The observed collective motion in heavy-ion collisions is a detailed probe of the created hot and dense matter. In non-central heavy-ion collisions the collective motion is azimuthally anisotropic due to the initial spatial anisotropic geometry of the overlap region and the pressure developed early in the collision. At RHIC energies the sizable azimuthal momentum-space anisotropy observed [1, 2] is the main evidence for the nearly perfect liquid behavior [3] of the created matter.

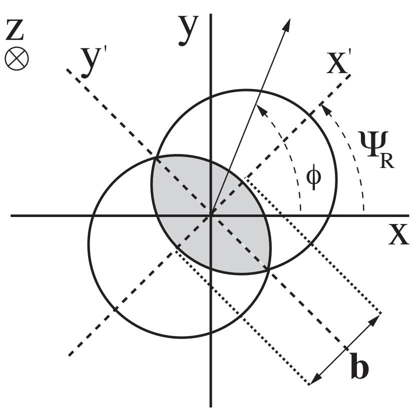

FIG. 1: Schematic view of a non-central nucleus-nucleus collision in the transverse plane.

The particle yield can be characterized by [4]:

$$E\frac{d^3N}{d^3\vec{p}} = \frac{1}{2\pi}\frac{d^2N}{p_t dp_t dy}\left(1+\sum_{n=1}^{\infty}2v_n\cos\left(n(\phi-\Psi_R)\right)\right), \quad (1)$$

where E is the energy of particle, $\vec{p}$ is particle's three-momentum, p_t is particle's transverse momentum, ϕ is its azimuthal angle, y is the rapidity and Ψ_R the reaction plane angle. The reaction plane angle, Ψ_R, is the azimuthal angle that impact parameter vector spans in the laboratory frame, see Fig 1. The Fourier coefficients v_n in Eq. (1) quantify the anisotropic flow. The first coefficient, v_1, of this Fourier series is called *directed flow* and second coefficient, v_2, is called *elliptic flow*. In general the coefficients v_n are p_t and y dependent—in this context we refer to them as *differential flow*. On the other hand when we are interested only in their average values we refer to them as *reference flow*. From Eq. (1) it follows that the average Fourier coefficients are given by:

$$\langle v_n\rangle = \langle\langle\cos n(\phi-\Psi_R)\rangle\rangle = \left\langle\left\langle e^{in(\phi-\Psi_R)}\right\rangle\right\rangle, \quad (2)$$

where the double brackets denote a statistical average over all particles and all events.

Since the reaction plane Ψ_R is not known experimentally, the anisotropic flow is estimated using azimuthal correlations between the observed particles. In the case of 2-particle azimuthal correlations the correlator is proportional to $\left\langle v_n^2\right\rangle$. This can be seen by:

$$\begin{aligned}\left\langle\left\langle e^{in(\phi_1-\phi_2)}\right\rangle\right\rangle &= \left\langle\left\langle e^{in(\phi_1-\Psi_R-(\phi_2-\Psi_R))}\right\rangle\right\rangle\\ &= \left\langle\left\langle e^{in(\phi_1-\Psi_R)}\right\rangle\left\langle e^{-in(\phi_2-\Psi_R)}\right\rangle\right\rangle\\ &= \left\langle v_n^2\right\rangle, \quad (3)\end{aligned}$$

where the double brackets denote a statistical average over all particles and all events (we keep this convention throughout the paper). Eq. (3) assumes that the azimuthal correlations between particles are only due to correlation of each particle with the reaction plane. To calculate azimuthal correlations it is customary to define a Q-vector evaluated in harmonic n:

$$Q_n \equiv \sum_{i=1}^{M} e^{in\phi_i}, \quad (4)$$

where M is the total number of reference particles in particular event.

Anisotropic flow is a collective effect because it originates from correlation of all the particles with the reaction plane. This can be exploited experimentally by using multiparticle correlations to estimate v_n. This approach, improved further to use *genuine* multiparticle correlations (cumulants), was adapted to measure anisotropic flow in [6, 7, 8] and it has the additional advantage that it allows to subtract systematically the so called nonflow effects from v_n. To calculate these multiparticle correlations directly the number of operations is approximately

$$\#\text{ of operations} \simeq \frac{\binom{M}{k}k!}{(\frac{k}{2})!(\frac{k}{2})!2!}\,, \tag{5}$$

where M is the number of particles in an event and $k = 2, 4, 6, ...$ for two, four, six, etc particle correlations. For the typical number of particles, of order 1000, measured in high energy heavy-ion collisions such an approach is clearly not feasible.

To overcome this problem the cumulants were instead expressed in terms of various moments of the modulus of the Q-vector [6]. The advantage is that the number of operations required is now $\propto M$ for each k. Unfortunately, flow estimates from cumulants constructed in such a way were systematically biased by the interference between various harmonics (this is clearly seen if one wants to estimate harmonic v_1 while v_2 is much larger).

An improved cumulant method using the formalism of generating functions was constructed [7, 8] which fixed the problem of interfering harmonics while keeping the number of operations $\propto M$ for each k. For this approach the calculation beyond $k = 2$ becomes analytically rather tedious and therefore the solutions are obtained using interpolation formulas. Unfortunately this introduces numerical uncertainties and requires tuning of interpolating parameters for different values of the flow harmonics v_n and multiplicity.

In this paper we present a new *analytic* way to calculate the cumulants. In our approach the cumulants are not biased by interference between various harmonics, interpolating formulas characteristic for the formalism of generating functions are not needed, and moreover we demonstrate that all detector effects can be analytically and automatically disentangled from the flow estimates in a single pass over the data. The number of operations required in our approach is still $\propto M$ for each k. Since in our approach cumulants are solely expressed in terms of expressions involving Q-vectors evaluated (in general) in different harmonics, we call them Q-*cumulants.*

A. Outline of the paper

II. SETTING UP THE STAGE

In this section we introduce the basic terminology and notation we will be using throughout the paper.

A. Multiparticle azimuthal correlations

For simplicity sake we consider explicitly only 2- and 4-particle azimuthal correlations—the generalization to azimuthal correlations involving more particles is straightforward and will not be presented here. In the way we define it, average 2- and 4-particle azimuthal correlations are obtained through the averaging procedure which consists of two distinct steps.

In the first step we define *single-event* average 2- and 4-particle azimuthal correlations in the following way:

$$\begin{aligned}
\langle 2\rangle &\equiv \left\langle e^{in(\phi_1-\phi_2)}\right\rangle \\
&\equiv \frac{1}{\binom{M}{2}2!}\sum_{\substack{i,j=1\\(i\neq j)}}^{M} e^{in(\phi_i-\phi_j)}\,, &(6)\\
\langle 4\rangle &\equiv \left\langle e^{in(\phi_1+\phi_2-\phi_3-\phi_4)}\right\rangle \\
&\equiv \frac{1}{\binom{M}{4}4!}\sum_{\substack{i,j,k,l=1\\(i\neq j\neq k\neq l)}}^{M} e^{in(\phi_i+\phi_j-\phi_k-\phi_l)}\,. &(7)
\end{aligned}$$

In above two equations ϕ_i is the azimuthal angle measured in laboratory frame of the i-th reference particle (in Section IV B we elaborate in detail on the most general case for labeling a particle as *Reference Particle* (RP) and/or *Particle Of Interest* (POI)). In order to avoid a trivial contribution coming from autocorrelations we have enforced the constraints $i \neq j$ and $i \neq j \neq k \neq l$ in Eqs. (6) and (7), respectively.

In the second step we define the final, *all-event* average 2- and 4-particle azimuthal correlations:

$$\begin{aligned}
\langle\langle 2\rangle\rangle &\equiv \left\langle\left\langle e^{in(\phi_1-\phi_2)}\right\rangle\right\rangle \\
&\equiv \frac{\sum_{i=1}^{N}(W_{\langle 2\rangle})_i\,\langle 2\rangle_i}{\sum_{i=1}^{N}(W_{\langle 2\rangle})_i}\,, &(8)\\
\langle\langle 4\rangle\rangle &\equiv \left\langle\left\langle e^{in(\phi_1+\phi_2-\phi_3-\phi_4)}\right\rangle\right\rangle \\
&\equiv \frac{\sum_{i=1}^{N}(W_{\langle 4\rangle})_i\,\langle 4\rangle_i}{\sum_{i=1}^{N}(W_{\langle 4\rangle})_i}\,, &(9)
\end{aligned}$$

where N is the number of events. In the second step we have introduced the event weights $W_{\langle 2\rangle}$ and $W_{\langle 4\rangle}$. They are determined event-by-event in terms of multiplicity in the following way:

$$\begin{aligned}
W_{\langle 2\rangle} &\equiv M(M-1)\,, &(10)\\
W_{\langle 4\rangle} &\equiv M(M-1)(M-2)(M-3)\,. &(11)
\end{aligned}$$

The above choice for the event weights reflects the number of different 2- and 4-particle combinations one can form for the event with multiplicity M. This choice of event weights minimizes the statistical spread.

B. Cumulants

In this section we briefly summarize the most important facts about cumulants. Consider first any two random observables x_1 and x_2 and their joint probability distribution function $f(x_1, x_2)$. In the case x_1 and x_2 are statistically independent the joint probability distribution function factorizes, namely

$$f(x_1, x_2) = f_{x_1}(x_1) f_{x_2}(x_2)\,. \tag{12}$$

In the case x_1 and x_2 are not statistically independent, i.e. if they are statistically *connected*, than there is a genuine 2-particle correlation in the system which we quantify in terms of genuine 2-particle probability distribution function $f_c(x_1, x_2)$. At the level of 2-particle correlation we have then in general the following decomposition:

$$f(x_1, x_2) = f_{x_1}(x_1) f_{x_2}(x_2) + f_c(x_1, x_2)\,. \tag{13}$$

Written in terms of expectation values this decomposition translates into:

$$E[x_1 x_2] = E[x_1]E[x_2] + E_c[x_1 x_2]\,. \tag{14}$$

In practice we rarely know the exact functional form of the p.d.f.'s appearing in Eq. (13). However, we can use measured (sampled) values of random variables x_1 and x_2 to construct an unbiased estimators for the expectation values in Eq. (14).

The last term on the RHS in Eq. (14), $E_c[x_1 x_2]$, is by definition the 2$^{\rm nd}$ order (or 2-particle) cumulant. It isolates the contribution to the expectation value $E[x_1 x_2]$ coming only from the genuine 2-particle correlation $f_c(x_1, x_2)$. This procedure can be generalized to any number of observables. Namely, for n random observables it is possible to isolate the contribution coming only from the genuine n-particle correlation, which is defined to be the $n^{\rm th}$ order (or n-particle) cumulant $E_c[x_1 x_2 \cdots x_n]$. The cumulant $E_c[x_1 x_2 \cdots x_n]$ is zero if one of the observables $x_1, x_2, \ldots, x_n$ is statistically independent from the others. Conversely, the cumulant $E_c[x_1 x_2 \cdots x_n]$ is not vanishing if and only if the variables $x_1, x_2, \ldots, x_n$ are statistically connected. For a proof of these statements we refer reader to [5].

1. *Cumulants in flow analysis*

The general formalism of cumulants highlighted in previous section was introduced into flow analysis practice by Ollitrault *et al* in [6, 7, 8]. As two random observables x_1 and x_2 Ollitrault *et al* have identified

$$\begin{aligned} x_1 &\equiv e^{in\phi_1}\,, \\ x_2 &\equiv e^{-in\phi_2}\,, \end{aligned} \tag{15}$$

where ϕ_1 and ϕ_2 are azimuthal angles of two particles measured in the laboratory frame. With this choice for random observables x_1 and x_2 we can write Eq. (14) in the following way:

$$E[e^{in(\phi_1-\phi_2)}] = E[e^{in\phi_1}]E[e^{-in\phi_2}] + E_c[e^{in(\phi_1-\phi_2)}]\,, \tag{16}$$

i.e.

$$E_c[e^{in(\phi_1-\phi_2)}] = E[e^{in(\phi_1-\phi_2)}] - E[e^{in\phi_1}]E[e^{-in\phi_2}]\,. \tag{17}$$

In practice we cannot estimate cumulant $E_c[e^{in(\phi_1-\phi_2)}]$ directly. Instead, we estimate from the sampled values of random observables ϕ_1 and ϕ_2 the quantities $E[e^{in\phi_1}]$, $E[e^{-in\phi_2}]$ and $E[e^{in(\phi_1-\phi_2)}]$, and than use the relation (17) to estimate the 2$^{\rm nd}$ order cumulant $E_c[e^{in(\phi_1-\phi_2)}]$. An immediate consequence of the choice (15) is that for the perfect detector (i.e. for detector with uniform azimuthal acceptance) $E[e^{in\phi_1}]$ and $E[e^{-in\phi_2}]$ vanish [7], so that we have

$$E_c[e^{in(\phi_1-\phi_2)}] = E[e^{in(\phi_1-\phi_2)}]\,. \tag{18}$$

An unbiased estimator for $E[e^{in(\phi_1-\phi_2)}]$ is an all-event average 2-particle correlation defined in Eq. (8), so that we finally have

$$c_n\{2\} = \langle\langle 2\rangle\rangle\,, \tag{19}$$

where we have used the standard notation $c_n\{2\}$ for the unbiased estimator of true 2$^{\rm nd}$ order cumulant $E_c[e^{in(\phi_1-\phi_2)}]$.

In order to get analogously the estimate for 4$^{\rm th}$ order cumulant we need to decompose the average 4-particle azimuthal correlation into its independent contributions. It was shown in [7] that for the case of detectors with uniform acceptance the average 4-particle correlation decomposes into:

$$\begin{aligned} E[e^{in(\phi_1+\phi_2-\phi_3-\phi_4)}] &= E[e^{in(\phi_1-\phi_3)}]E[e^{in(\phi_2-\phi_4)}] \\ &+ E[e^{in(\phi_1-\phi_4)}]E[e^{in(\phi_2-\phi_3)}] \\ &+ E_c[e^{in(\phi_1+\phi_2-\phi_3-\phi_4)}]\,, \end{aligned} \tag{20}$$

which we can invert to isolate the average *genuine* 4-particle correlation $E_c[e^{in(\phi_1+\phi_2-\phi_3-\phi_4)}]$, which is by definition the 4$^{\rm th}$ order (or 4-particle) cumulant:

$$\begin{aligned} E_c[e^{in(\phi_1+\phi_2-\phi_3-\phi_4)}] &= E[e^{in(\phi_1+\phi_2-\phi_3-\phi_4)}] \\ &- E[e^{in(\phi_1-\phi_3)}]E[e^{in(\phi_2-\phi_4)}] \\ &- E[e^{in(\phi_1-\phi_4)}]E[e^{in(\phi_2-\phi_3)}]\,. \end{aligned} \tag{21}$$

Taking into account the unbiased estimators for 4- and 2-particle correlations given in Eqs. (9) and (8), respectively, the relation (21) translates into:

$$c_n\{4\} = \langle\langle 4\rangle\rangle - 2\cdot\langle\langle 2\rangle\rangle^2\,, \tag{22}$$

where now $c_n\{4\}$ stands for the unbiased estimator of true 4$^{\rm th}$ order cumulant $E_c[e^{in(\phi_1+\phi_2-\phi_3-\phi_4)}]$. We remark that expressions (19) and (22) are applicable only for detectors with uniform acceptance and will be generalized in Appendix C to extend their applicability also for the detectors with non-uniform acceptance.

The physical significance of cumulants lies in the fact that all particles produced in a heavy-ion collision are correlated to the reaction plane characterizing the geometry of that collision (so called *flow correlations*). The correlation of each particle to the reaction plane induces the genuine multiparticle correlation for any number of particles in the correlator. On the other hand, multiparticle correlations which do not originate from the correlation of each particle to the reaction plane (so called *non-flow correlations*) typically involve only few particles—this mean that they will contribute only to correlators involving few particles. As an example, if there is a genuine 2-particle nonflow correlation in the system (originating for instance from the 2-particle resonance decays or from track splitting in the detector), than $2^{\rm nd}$ order cumulant is sensitive to it while $4^{\rm th}$ order cumulant is not. Usually the genuine 2-particle nonflow correlation is denoted by δ_2 and for an event with multiplicity M its strength can be roughly estimated as:

$$\delta_2 \sim \frac{1}{M-1}\,, \tag{23}$$

simply because this is the probability that once we have specified the first particle in the correlator we will out of the remaining $M-1$ particles pick up as the second particle in the correlator the one which is correlated to the first chosen particle in the correlator. On the other hand, if there are flow correlations in the system, than they will induce contribution to genuine multiparticle correlations for any number of particles in the correlator and both $2^{\rm nd}$ and $4^{\rm th}$ order cumulant will be sensitive to them. Hence in this example only the $4^{\rm th}$ order cumulant can disentangle the contribution coming from flow correlations from the contribution coming from genuine 2-particle nonflow correlations (this will be illustrated more explicitly with the example in Section III C).

Having obtained estimates for cumulants one can easily use them to estimate reference flow harmonics. Different order cumulants provide independent estimates for the same reference harmonic v_n. In particular [7],

$$\begin{aligned} v_n\{2\} &= \sqrt{c_n\{2\}}\,, & (24)\\ v_n\{4\} &= \sqrt[4]{-c_n\{4\}}\,, & (25)\end{aligned}$$

where notation $v_n\{2\}$ was used to denote the reference flow harmonic v_n estimated from the $2^{\rm nd}$ order cumulant $c_n\{2\}$ and $v_n\{4\}$ stands for the reference flow harmonic v_n estimated from the $4^{\rm th}$ order cumulant $c_n\{4\}$.

After we have introduced in this section notation and terminology in subsequent sections we present our results.

III. ESTIMATING REFERENCE FLOW HARMONICS FROM Q-CUMULANTS

As already indicated in the introduction it is possible analytically to express cumulants solely in terms of expressions consisting of Q-vectors evaluated (in general) in different harmonics. In order to distinguish these cumulants from the cumulants calculated with formalism of generating functions we call them Q-cumulants. First we outline Q-cumulants that shall be used to estimate reference flow harmonics for the case of detector with uniform azimuthal acceptance.

A. $2^{\rm nd}$ order

To obtain $2^{\rm nd}$ order Q-cumulant it suffices to decompose $|Q_n|^2$ which is straight from the definition (4) given by

$$|Q_n|^2 = \sum_{i,j=1}^{M} e^{in(\phi_i-\phi_j)}\,. \tag{26}$$

It is clear that the two summing indices i and j can be either the same or different in the above relation. Physically, when the indices are different we are correlating two different particles and when the indices are the same we are correlating particle to itself (autocorrelation). It follows that in the decomposition of $|Q_n|^2$ we have 2-particle and 1-particle contributions with the following combinatorial coefficients:

$$\begin{aligned} &2\mathrm{-particle} \;:\; \langle 2\rangle \cdot \binom{M}{2} 2!\,,\\ &1\mathrm{-particle} \;:\; 1\cdot M\,, \end{aligned} \tag{27}$$

where $\langle 2\rangle$ was defined in Eq. (6). Written explicitly,

$$|Q_n|^2 = \langle 2\rangle \cdot \binom{M}{2} 2! + 1\cdot M\,, \tag{28}$$

which can be trivially solved to obtain $\langle 2\rangle$. If follows that the single-event average 2-particle azimuthal correlation is analytically given by the following equation:

$$\langle 2\rangle = \frac{|Q_n|^2 - M}{M(M-1)}\,, \tag{29}$$

where Q_n is the Q-vector defined in Eq. (4) and M is the multiplicity of an event. To get the final, all-event average 2-particle azimuthal correlation $\langle\langle 2\rangle\rangle$ one has to use definitions (8) and (10). By making use of Eq. (19) the $2^{\rm nd}$ order Q-cumulant is than simply

$$c_n\{2\} = \frac{\sum_{i=1}^{N}\left(|Q_n|_i^2 - M_i\right)}{\sum_{i=1}^{N} M_i(M_i-1)}\,. \tag{30}$$

In above equation $|Q_n|_i$ is the modulus of Q-vector (4) in the i-th event, M_i is the number of reference particles in the i-th event and N is total number of events. Eq. (30) is the analytic expression for 2^{nd} order Q-cumulant and its evaluation requires a single loop over data. In Appendix C the Eq. (30) will be further generalized to account for detector effects.

B. 4^{th} order

This section we start by decomposing $|Q_n|^4$. From definition of Q-vector (4) it follows that

$$|Q_n|^4 = \sum_{i,j,k,l=1}^{M} e^{in(\phi_i+\phi_j-\phi_k-\phi_l)} \,. \tag{31}$$

In the above summation we can have four distinct cases for the indices i, j, k and l—either they are all different (4-particle correlation), either there are three different indices, either there are two different indices, or all indices are the same (autocorrelation). Very important thing to note, however, is that for instance for the case of three different indices we can end up either with 3-particle correlation if the two indices which are the same are on the same side of the correlator (e.g. $\phi_i+\phi_i-\phi_k-\phi_l = 2\phi_i-\phi_k-\phi_l$) or with the 2-particle correlation if the two indices which are the same are on the opposite side of the correlator (e.g. $\phi_i+\phi_j-\phi_k-\phi_i = \phi_j-\phi_k$). Having this in mind we have obtained the following analytic result for the single-event average 4-particle correlation defined in Eq. (7):

$$\begin{aligned}\langle 4\rangle &= \frac{|Q_n|^4+|Q_{2n}|^2-2\cdot\mathfrak{Re}\left[Q_{2n}Q_n^*Q_n^*\right]}{M(M-1)(M-2)(M-3)}\\ &- 2\,\frac{2(M-2)\cdot|Q_n|^2-M(M-3)}{M(M-1)(M-2)(M-3)}\,.\end{aligned} \tag{32}$$

The detailed derivation of this result is presented in Appendix A. The final all-event average 4-particle azimuthal correlation, $\langle\langle 4\rangle\rangle$, is then obtained by making use of Eqs. (9) and (11). Having obtained results for $\langle\langle 4\rangle\rangle$ and $\langle\langle 2\rangle\rangle$ we can now calculate the 4^{th} order Q-cumulant from Eq. (22). It follows:

$$\begin{aligned}c_n\{4\} &= \frac{\sum_{i=1}^{N}\left(|Q_n|_i^4+|Q_{2n}|_i^2-2\cdot\mathfrak{Re}\left[Q_{2n}Q_n^*Q_n^*\right]_i\right)}{\sum_{i=1}^{N}M_i(M_i-1)(M_i-2)(M_i-3)}\\ &- 2\,\frac{\sum_{i=1}^{N}\left(2(M_i-2)\cdot|Q_n|_i^2-M_i(M_i-3)\right)}{\sum_{i=1}^{N}M_i(M_i-1)(M_i-2)(M_i-3)}\\ &- 2\left[\frac{\sum_{i=1}^{N}\left(|Q_n|_i^2-M_i\right)}{\sum_{i=1}^{N}M_i(M_i-1)}\right]^2\,.\end{aligned} \tag{33}$$

In above expression $|Q_n|_i$ and $|Q_{2n}|_i$ are moduli of Q-vector (4) evaluated in harmonics n and $2n$, respectively, in the i-th event, $\mathfrak{Re}\left[Q_{2n}Q_n^*Q_n^*\right]_i$ stands for the real part of product $Q_{2n}Q_n^*Q_n^*$ in the i-th event, M_i is the multiplicity of i-th event and N is the total number of events. The result (30) is the analytic result for 4^{th} order Q-cumulant and its evaluation requires a single loop over data. In Appendix C this result will be further generalized in order to extend its applicability also for the detectors with non-uniform acceptance.

In the next section with few examples we illustrate the performance of Q-cumulants in estimating reference flow harmonics and we point out the improvements over the cumulants proposed by Ollitrault *et al* [6, 7, 8] which were used so far in the flow analysis.

C. Examples

In this section with the series of plots we illustrate the performance of Q-cumulants. In each plot bellow in the first bin we place the Monte Carlo estimate for v_n and denote it by $v_n\{\text{MC}\}$. To this estimate the estimates for v_n obtained from different order Q-cumulants are being compared. Estimate for v_n obtained from the k^{th} order Q-cumulant is denoted by $v_n\{k,\text{QC}\}$, where $k=2,4,6,8$. The grey mesh in each plot is spanned by the error bars of the Monte Carlo estimate $v_n\{\text{MC}\}$.

a. Interference between harmonics. As already mentioned in the introduction the first cumulants proposed by Ollitrault *et al* [6] were systematically biased by the interference between harmonics. On the other hand the Q-cumulants can completely disentangle this interference and this can be clearly seen in examples presented in Figs. 2, 17, 18, 19, 20. The reason why the original cumulants proposed by Ollitrault *et al* were biased lies in the fact that they have omitted the terms which consist of Q-vectors evaluated in *different* harmonics (for instance terms $|Q_{2n}|^2$ and $\mathfrak{Re}\left[Q_{2n}Q_n^*Q_n^*\right]$ in Eq. (33)—such terms do appear in the analytic results and are crucial in disentangling the interference between harmonics).

b. Numerical stability. In the second paper on cumulants [7, 8] Ollitrault *et al* proposed usage of generating function accompanied with the prescription of interpolating procedure to be used to calculate cumulants in practice. This proposal improved over their first proposal [6] in a sense that the issue with interfering harmonics have been resolved here. However, the limitation of their new approach lies in the fact that for different values of flow harmonics and multiplicity one has to tune in a different way parameters on which the interpolating procedure relies (see in particular Eq. (5) in [8] and comments following this equation). This limitation is not present in the case of Q-cumulants for which the very same analytic equations can be used for very different values of flow harmonics and multiplicity. This is illustrated with example in Fig. 3.

c. Nonflow. We have already shown in introduction in Eq. (3) that when only flow correlations are present

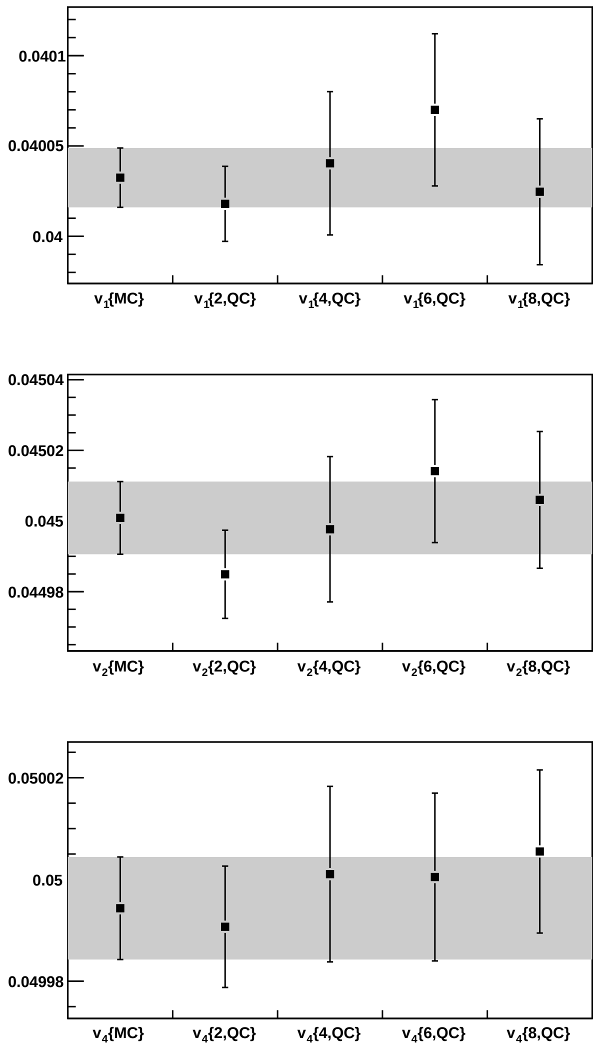

FIG. 2: Particle azimuthal angles were sampled from azimuthal distribution (1) characterized with a presence of three harmonics, namely $v_1 = 0.04$, $v_2 = 0.045$ and $v_4 = 0.05$. On the top plot are estimates for harmonic v_1, in the middle plot are estimates for harmonic v_2 and on the bottom plot are estimates for harmonic v_4. Each harmonic can be correctly estimated with Q-cumulants, the presence of other two harmonics is completely disentangled.

we have for the case of detector with uniform acceptance:

$$\langle\langle 2\rangle\rangle = \left\langle v_n^2\right\rangle \,. \qquad (34)$$

By following the same line of reasoning as used in deriving the Eq. (3) it follows straightforwardly:

$$\langle\langle 4\rangle\rangle = \left\langle v_n^4\right\rangle \,. \qquad (35)$$

For the sake of simplicity of the argument to be presented now in the remainder of this section we shall assume that statistical flow fluctuations are negligible, so that we can write

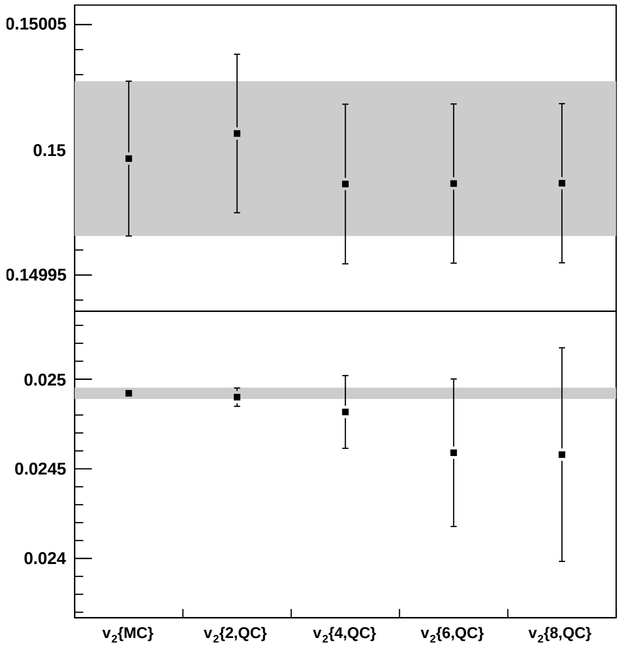

FIG. 3: Per event 500 particles were sampled from azimuthal distribution (1) characterized with harmonic $v_2 = 15\%$ (*top*) and $v_2 = 2.5\%$ (*bottom*) in the total of 10^6 events. The very same analytic equations for Q-cumulants were used to estimate successfully the flow harmonic v_2 in both cases.

$$\langle\langle 2\rangle\rangle \simeq \langle v_n\rangle^2 \,, \qquad (36)$$
$$\langle\langle 4\rangle\rangle \simeq \langle v_n\rangle^4 \,. \qquad (37)$$

When it comes to cumulants, for the case of detector with uniform acceptance we have from previous sections that $c_n\{2\} = \langle\langle 2\rangle\rangle$, so the estimates for v_n coming from $c_n\{2\}$ and $\langle\langle 2\rangle\rangle$ will be represented with the same result $v_n\{2,\mathrm{QC}\}$ in the plots bellow. After plugging results (36) and (37) into (22) it follows:

$$c_n\{4\} \simeq -\langle v_n\rangle^4 \,. \qquad (38)$$

Therefore when only flow correlations are present we can use all of quantities $\langle\langle 2\rangle\rangle$, $c_n\{2\}$, $\langle\langle 4\rangle\rangle$ and $c_n\{4\}$ to estimate correctly the flow harmonics:

$$\begin{aligned} v_n\{2,\mathrm{QC}\} &\equiv \sqrt{c_n\{2\}} \equiv \sqrt{\langle\langle 2\rangle\rangle} \simeq \langle v_n\rangle \,, \\ v_n\{4,\langle\langle 4\rangle\rangle\} &\equiv \sqrt[4]{\langle\langle 4\rangle\rangle} \simeq \langle v_n\rangle \,, \\ v_n\{4,\mathrm{QC}\} &\equiv \sqrt[4]{-c_n\{4\}} \simeq \langle v_n\rangle \,. \end{aligned} \qquad (39)$$

This is illustrated in Fig. 4 (top).

Now we shall use a simple qualitative model to add genuine 2-particle nonflow correlations in the system—each produced particle will be taken twice in the analysis (this might correspond in reality to track splitting due to the inefficiencies in the detector.) For original sample of $M/2$ particles we build up the Q-vector and estimate the flow harmonic in the following way:

$$\langle 2\rangle = \frac{|Q_n|^2-\frac{M}{2}}{\frac{M}{2}(\frac{M}{2}-1)} = v_n^2\,. \tag{40}$$

For the new sample of M particles in which each particle is taken twice we have the following result:

$$\begin{aligned}\langle 2\rangle &= \frac{|2\cdot Q_n|^2-M}{M(M-1)}\\ &= \frac{|Q_n|^2-\frac{M}{2}}{\frac{M}{2}(\frac{M}{2}-1)} + \frac{M^2-4\,|Q_n|^2}{M(M-1)(M-2)}\\ &= v_n^2 + \frac{M^2-4\left(v_n^2\frac{M}{2}\left(\frac{M}{2}-1\right)+\frac{M}{2}\right)}{M(M-1)(M-2)}\\ &= v_n^2+\frac{1-v_n^2}{M-1}\,,\end{aligned} \tag{41}$$

where in obtaining the 3rd line we have inserted result (40). Since $v_n^2 \ll 1$, assuming for simplicity that statistical flow fluctuations are negligible and that multiplicity is the same in each event, we have

$$\langle\langle 2\rangle\rangle \simeq \langle v_n\rangle^2 + \frac{1}{M-1}\,, \tag{42}$$

from which we can read off that the genuine 2-particle nonflow contribution to the average 2-particle correlation is

$$\delta_2 \simeq \frac{1}{M-1}\,, \tag{43}$$

as already advocated in Section II B 1 based purely on combinatorial grounds.

By performing the same calculation for $\langle 4\rangle$ given in Eq. (32), using $v_n \ll 1$ and taking for simplicity $v_{2n}=0$, we have obtained for the sample consisting both of original and doubled tracks the following result:

$$\begin{aligned}\langle 4\rangle &\simeq v_n^4 \\ &+2\frac{36-22M+M^3+Mv_n^2(2M^3-10M^2+17M-24)}{(M-1)(M-2)(M-3)(M-4)(M-6)}\,.\end{aligned} \tag{44}$$

To make a further progress we shall assume that statistical flow fluctuations are negligible and that multiplicity is large and constant in each event. Under these assumptions and taking into account Eq. (42) and Eq. (44) we arrive at our final results for average 2- and 4-particle correlations in a sample with doubled tracks:

$$\langle\langle 2\rangle\rangle \simeq \langle v_n\rangle^2 + \frac{1}{M}\,, \tag{45}$$

$$\langle\langle 4\rangle\rangle \simeq \langle v_n\rangle^4 + 2\,\frac{1+2v_n^2M}{M^2}\,. \tag{46}$$

This leads us to the following results for flow estimates:

$$v_n\{2,\mathrm{QC}\} \simeq \sqrt{\langle v_n\rangle^2+\frac{1}{M}}\,, \tag{47}$$

$$v_n\{4,\langle\langle 4\rangle\rangle\} \simeq \sqrt[4]{\langle v_n\rangle^4+\frac{4\,\langle v_n\rangle^2}{M}+\frac{2}{M^2}}\,, \tag{48}$$

which indicate that in the case of track splitting the flow estimates from the 2nd order Q-cumulant and from the average 4-particle correlation will be strongly biased.

On the other hand, from Eq. (22) and from results (45) and (46) it follows that the 4th order cumulant, i.e. the genuine 4-particle correlation, is:

$$\begin{aligned}c_n\{4\} &\simeq \langle v_n\rangle^4 + 2\,\frac{1+2v_n^2M}{M^2} - 2\cdot\left(\langle v_n\rangle^2+\frac{1}{M}\right)^2\\ &= -\langle v_n\rangle^4\,,\end{aligned} \tag{49}$$

meaning that it is not sensitive to the effects of track splitting, so that still we have

$$v_n\{4,\mathrm{QC}\} \equiv \sqrt[4]{-c_n\{4\}} \simeq \langle v_n\rangle\,. \tag{50}$$

The results (47), (48) and (50) obtained from simple qualitative model illustrate that in the case genuine 2-particle nonflow correlations are present in the system only the 4th and higher order cumulants still provide an unbiased estimate for the flow harmonics. This is further illustrated in Fig. 4 (bottom). Analogously one can show that in the case genuine 4-particle nonflow correlations are present in the system only the 6th and higher order cumulants are not sensitive to them—lower order nonflow contribution is systematically removed by the higher order cumulants.

d. Detector effects. There are two main approaches to correct for detector inefficiencies in flow analysis. For the methods based on expressions involving Q-vectors one usually performs a separate run over data to obtain the azimuthal profile of detector's acceptance. Once obtained, this azimuthal profile is inverted and normalized to get so called w_ϕ-weights. In the subsequent runs over data when building a Q-vector each particle is weighted with the w_ϕ-weight corresponding to the azimuthal bin to which its azimuthal angle belongs. This technique (sometimes also called *flattening*) is presented in Appendix B. The limitation of this technique lies in the fact that if detector has a gap in azimuthal acceptance its azimuthal profile cannot be inverted, and correspondingly the w_ϕ-weights cannot be obtained.

Methods for flow analysis based on the formalism of generating functions relies on the pragmatic approach to correct for detector effects. Namely, at the level of generating function on makes few projections on a fixed, equally spaced, directions in the detector and for each direction obtains independent estimates for flow harmonics. The final estimate for flow harmonics is than obtained as average over all those direction-wise flow estimates.

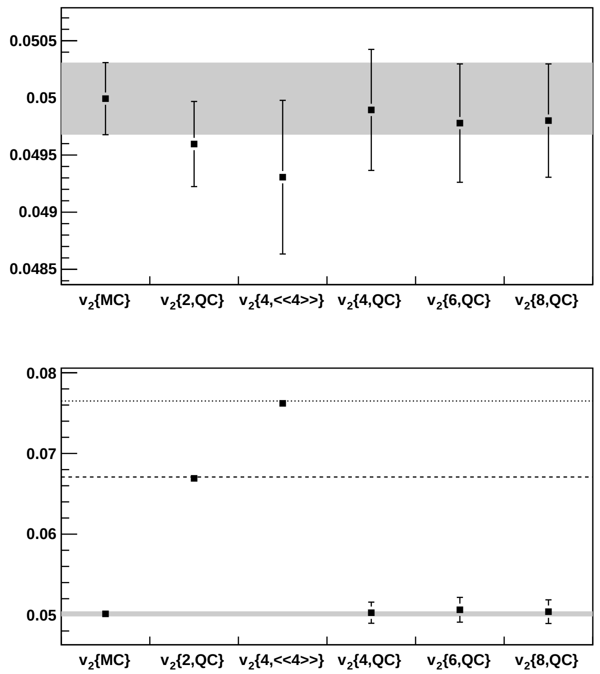

FIG. 4: (*top*) In this example 500 particles were sampled per event from azimuthal distribution (1) characterized with harmonic $v_2 = 0.05$. Only flow correlations are present in the system and both the multiparticle estimates and estimates from Q-cumulants are estimating v_2 correctly. (*bottom*) In this example 250 particles were sampled per event from the same azimuthal distribution as used in the top plot but each sampled particle was taken twice in order to simulate strong 2-particle nonflow correlations. With dashed line is indicated theoretical result from Eqs. (47) and with dotted line from Eq. (48). Only higher order cumulants can provide correct estimate for v_2 in the case genuine 2-particle nonflow correlations are present. (In both examples total number of events is $N = 10^5$.)

Although pragmatic, this approach actually works quite well in practice.

When it comes to Q-cumulants, it is possible analytically to account for detector effects. We recall Eq. (17) which gives that the generalized 2$^{\rm nd}$ order Q-cumulant which can be also used for detectors with non-uniform acceptance is:

$$
\begin{aligned}
c_n\{2\} &= \langle\langle 2\rangle\rangle - \mathfrak{Re}\Big\{ \left[\, \langle\langle \cos n\phi_1\rangle\rangle + i\,\langle\langle \sin n\phi_1\rangle\rangle \,\right] \\
&\qquad \times \left[\, \langle\langle \cos n\phi_2\rangle\rangle - i\,\langle\langle \sin n\phi_2\rangle\rangle \,\right] \Big\} \\
&= \langle\langle 2\rangle\rangle - \langle\langle \cos n\phi_1\rangle\rangle^2 - \langle\langle \sin n\phi_1\rangle\rangle^2 \,, \qquad (51)
\end{aligned}
$$

where in obtaining the last line we have used the fact that for instance $\langle\langle \cos n\phi_1\rangle\rangle$ and $\langle\langle \cos n\phi_2\rangle\rangle$ are the same quantities apart from the trivial relabeling. Remarkably, only two additional terms appear in Eq. (51), namely $\langle\langle \cos n\phi_1\rangle\rangle^2$ and $\langle\langle \sin n\phi_1\rangle\rangle^2$, which counterbalance the bias to $\langle\langle 2\rangle\rangle$ coming from the very general type of detector inefficiencies so that $c_n\{2\}$ remains unbiased. Further details and results are outlined in Appendix C, here we just present two illustrative examples on Figs. 21 and 6.

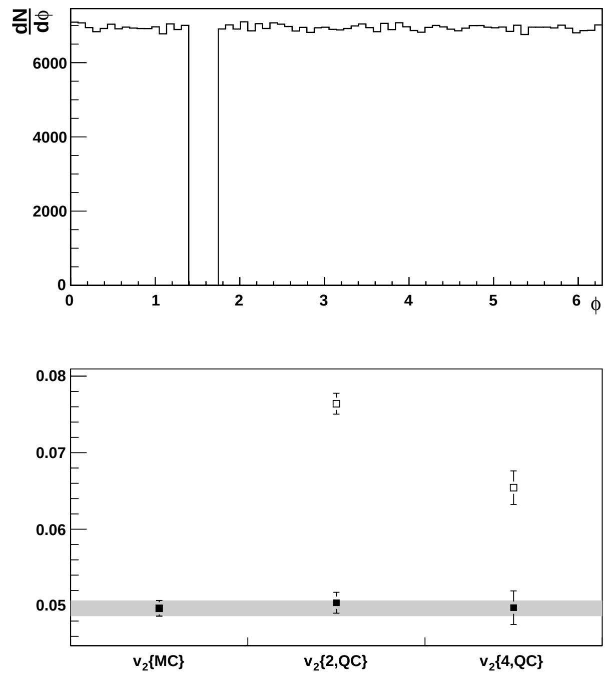

FIG. 5: Example for small non-uniform acceptance when particles emitted in $80^{\rm o} \leq \phi < 100^{\rm o}$ are blocked. Per event 500 particles were sampled from azimuthal distribution (1) characterized with $v_2 = 0.05$ in total number of 1000 events. Detector azimuthal profile is shown on the top plot. On the bottom plot with open markers are shown estimates from "isotropic" cumulants (defined in Eqs. (19) and (22)) and with closed markers estimates from "generalized" cumulants (Eqs. (C1) and (C6)). Bias due to detector defects is clearly not negligible (open markers) and has to be corrected for thoroughly (closed markers).

IV. ESTIMATING DIFFERENTIAL FLOW FROM Q-CUMULANTS

Multiparticle correlations introduced in the previous section were used to estimate cumulants which were than used to estimate reference flow. However, in order to estimate the differential flow of particles of interest with respect to this reference flow we need *reduced* multiparticle azimuthal correlations. By reduced multiparticle azimuthal correlations we mean the multiparticle azimuthal correlations obtained after restricting one particle in correlator to belong only to the phase window of interest. In

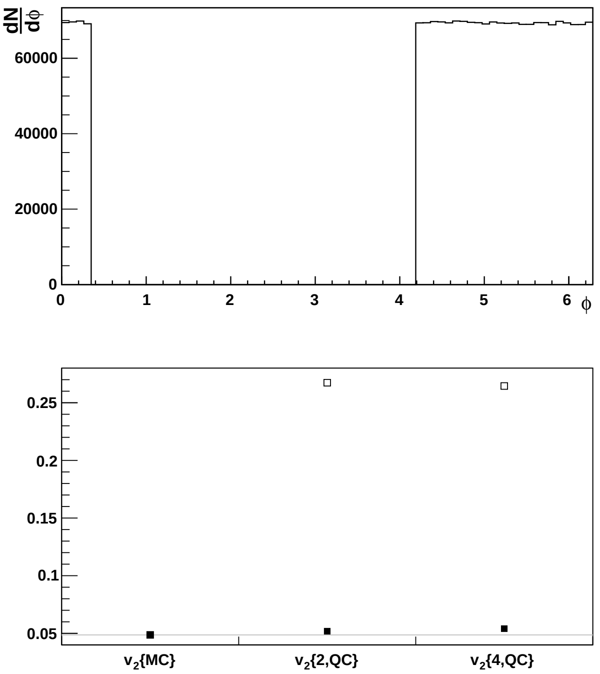

FIG. 6: Example for the huge non-uniform acceptance when particles emitted in $20^o \leq \phi < 240^o$ were blocked. Per event 500 particles were sampled from azimuthal distribution (1) characterized with $v_2 = 0.05$ in total number of 1000 events. Detector azimuthal profile is shown on the top plot. On the bottom plot with open markers are shown estimates from “isotropic” cumulants (defined in Eqs. (19) and (22)) and with closed markers estimates from “generalized” cumulants (Eqs. (C1) and (C6)). Systematic bias due to detector defects is clearly huge (open markers) but with “generalized” cumulants can be corrected for (closed markers).

practice the phase window of interest can comprise particular p_t or y bin, subset of all particles consisting only of identified particles (pions, protons, kaons), and so on. But first we outline the most general cases for particle labeling.

A. Particle labels

For particles selected for flow analysis we use two labels, namely RP (*Reference Particle*) and/or POI (*Particle Of Interest*). These labels are needed because flow analysis is being performed in two distinct steps. In the first step we estimate the reference flow by using only the RPs, while in the second step we estimate the differential flow of POIs with respect to the reference flow of RPs obtained in the first step. In practice there are three distinct cases for particle labeling:

1. *no overlap*: Neither particle labeled as POI was also labeled as RP;
2. *partial overlap*: Some particles labeled as POI were also labeled as RP;
3. *full overlap*: All particles labeled as POI were also labeled as RP.

Later in the paper we provide formulas which automatically handle all three distinct cases.

B. Reduced multiparticle azimuthal correlations

For reduced *single-event* average 2- and 4-particle azimuthal correlations we use the following notations and definitions:

$$\begin{aligned}\langle 2'\rangle &\equiv \left\langle e^{in(\psi_1-\phi_2)}\right\rangle \\ &\equiv \frac{1}{m_pM-m_q}\sum_{i=1}^{m_p}\sum_{\substack{j=1\\(i\neq j)}}^{M} e^{in(\psi_i-\phi_j)}\,, \qquad (52)\\ \langle 4'\rangle &\equiv \left\langle e^{in(\psi_1+\phi_2-\phi_3-\phi_4)}\right\rangle \\ &\equiv \frac{1}{(m_pM-3m_q)(M-1)(M-2)} \\ &\times \sum_{i=1}^{m_p}\sum_{\substack{j,k,l=1\\(i\neq j\neq k\neq l)}}^{M} e^{in(\psi_i+\phi_j-\phi_k-\phi_l)}\,, \qquad (53)\end{aligned}$$

where m_p is the total number of particles labeled as POI (some of which might have been also labeled additionally as RP) in a phase window of interest, m_q is the total number of particles labeled *both* as RP and POI in a phase window of interest, M is the total number of particles labeled as RP (some of which might have been also labeled additionally as POI) in the *whole* event, ψ_i is the azimuthal angle of the i-th particle labeled as POI and taken from a phase window of interest (taken even if it was also additionally labeled as RP), ϕ_j is the azimuthal angle of the j-th particle labeled as RP and taken from the whole event (taken even if it was also additionally labeled as POI).

On the other hand the final, *all-event*, average reduced 2- and 4-particle correlations we denote and define in the following way:

$$\langle\langle 2'\rangle\rangle \equiv \frac{\sum_{i=1}^{N}(W_{\langle 2'\rangle})_i\,\langle 2'\rangle_i}{\sum_{i=1}^{N}(W_{\langle 2'\rangle})_i}\,, \qquad (54)$$

$$\langle\langle 4'\rangle\rangle \equiv \frac{\sum_{i=1}^{N}(W_{\langle 4'\rangle})_i\,\langle 4'\rangle_i}{\sum_{i=1}^{N}(W_{\langle 4'\rangle})_i}\,, \qquad (55)$$

where N is the number of events. In the second step we have introduced the event weights $W_{\langle 2'\rangle}$ and $W_{\langle 4'\rangle}$ for single-event average reduced 2- and 4-particle correlations (52) and (53), respectively. They are determined event-by-event in the following way:

$$W_{\langle 2'\rangle} \equiv m_pM-m_q\,, \qquad (56)$$

$$W_{\langle 4'\rangle} \equiv (m_pM-3m_q)(M-1)(M-2)\,, \qquad (57)$$

where meaning of m_p, m_q and M was given in the text following Eq. (53).

C. p- and q-vectors

In order to express our subsequent results in a more elegant way, we will in analogy with the Q-vector for the whole event defined in Eq. (4) now introduce p- and q-vectors for the phase window of interest.

Taking all POIs (m_p in total) from the phase window of interest within particular event we build up the following quantity for that phase window:

$$p_n \equiv \sum_{i=1}^{m_p} e^{in\psi_i} . \tag{58}$$

Furthermore, for the subset of POIs which consists of all particles from the phase window of interest within particular event labeled *both* as POI and RP (m_q in total) we introduce

$$q_n \equiv \sum_{i=1}^{m_q} e^{in\psi_i} . \tag{59}$$

Having introduced some additional quantities in this section we are now ready to present our analytic results for the Q-cumulants that shall be used to estimate differential flow harmonics (for convenience sake we refer to them as *differential Q-cumulants*).

D. Differential Q-cumulants

In this section we present the results for the differential Q-cumulants and outline the prescription for estimating differential flow harmonics.

1. 2^{nd} *order*

By making use of terminology and notation introduced in previous sections we have obtained the following analytic result for the average reduced single- and all-event 2-particle correlations:

$$\langle 2' \rangle = \frac{p_n Q_n^* - m_q}{m_p M - m_q} , \tag{60}$$

$$\langle\langle 2' \rangle\rangle = \frac{\sum_{i=1}^N (w_{\langle 2' \rangle})_i \langle 2' \rangle_i}{\sum_{i=1}^N (w_{\langle 2' \rangle})_i} , \tag{61}$$

$$w_{\langle 2' \rangle} = m_p M - m_q . \tag{62}$$

In the spirit of Ollitrault *et al* [7] we now denote by $d_n\{2\}$ the 2^{nd} order differential Q-cumulant that shall be used to estimate differential flow and estimate it for detectors with uniform azimuthal acceptance as

$$d_n\{2\} = \langle\langle 2' \rangle\rangle . \tag{63}$$

Taking into account Eqs. (60–62) it follows:

$$d_n\{2\} = \frac{\sum_{i=1}^N (p_n Q_n^* - m_q)_i}{\sum_{i=1}^N (m_p M - m_q)_i} . \tag{64}$$

This is the analytic results and it will be generalized in Appendix C to extend its applicability also for the detectors with non-uniform acceptance. Estimates of differential flow harmonics v_n' are being denoted as $v_n'\{2\}$ and are given by [7]:

$$v_n'\{2\} = \frac{d_n\{2\}}{\sqrt{c_n\{2\}}} , \tag{65}$$

where the analytic expression for $d_n\{2\}$ and $c_n\{2\}$ are given in Eqs. (64) and (30), respectively.

2. 4^{th} *order*

At the level of 4-particle correlations, we have obtained the following analytic results for single- and all-event reduced 4-particle correlations:

$$\begin{aligned}
\langle 4' \rangle = \Big[& p_n Q_n Q_n^* Q_n^* - q_{2n} Q_n^* Q_n^* - p_n Q_n Q_{2n}^* \\
- & 2 \cdot M p_n Q_n^* - 2 \cdot m_q |Q_n|^2 + 7 \cdot q_n Q_n^* \\
- & Q_n q_n^* + q_{2n} Q_{2n}^* + 2 \cdot p_n Q_n^* \\
+ & 2 \cdot m_q M - 6 \cdot m_q \Big] \\
/ & \Big[(m_p M - 3 m_q)(M-1)(M-2) \Big] ,
\end{aligned} \tag{66}$$

$$\langle\langle 4' \rangle\rangle = \frac{\sum_{i=1}^N (w_{\langle 4' \rangle})_i \langle 4' \rangle_i}{\sum_{i=1}^N (w_{\langle 4' \rangle})_i} , \tag{67}$$

$$w_{\langle 4' \rangle} = (m_p M - 3 m_q)(M-1)(M-2) . \tag{68}$$

By following prescription outlined by Ollitrault *et al* in [7] we now denote by $d_n\{4\}$ the 4^{th} order differential Q-cumulant and estimate it by

$$d_n\{4\} = \langle\langle 4' \rangle\rangle - 2 \cdot \langle\langle 2' \rangle\rangle \langle\langle 2 \rangle\rangle . \tag{69}$$

Plugging results for $\langle\langle 4'\rangle\rangle$, $\langle\langle 2'\rangle\rangle$ and $\langle\langle 2\rangle\rangle$ into Eq. (69) we arrive at the following result:

$$
\begin{aligned}
d_n\{4\} &= \Bigg\{\sum_{i=1}^{N}\Big[p_nQ_nQ_n^*Q_n^* - q_{2n}Q_n^*Q_n^* - p_nQ_nQ_{2n}^* \\
&- 2\cdot Mp_nQ_n^* - 2\cdot m_q\,|Q_n|^2 + 7\cdot q_nQ_n^* - Q_nq_n^* \\
&+ q_{2n}Q_{2n}^* + 2\cdot p_nQ_n^* + 2\cdot m_qM - 6\cdot m_q\Big]_i \\
&/ \sum_{i=1}^{N}\Big[(m_pM-3m_q)(M-1)(M-2)\Big]_i\Bigg\} \\
&- 2\cdot\frac{\sum_{i=1}^{N}(p_nQ_n^*-m_q)_i}{\sum_{i=1}^{N}(m_pM-m_q)_i} \\
&\times \frac{\sum_{i=1}^{N}\left(|Q_n|^2-M\right)_i}{\sum_{i=1}^{N}(M(M-1))_i}\,. \qquad (70)
\end{aligned}
$$

This is an analytic result for 4th order differential Q-cumulant and it is applicable only for the detectors with uniform azimuthal acceptance—more general result which can be used also for detectors with non-uniform acceptance is outlined in Appendix C. Having obtained estimates for $d_n\{4\}$ and $c_n\{4\}$ in Eqs. (70) and (33), respectively, we can estimate differential flow harmonics v_n' in the following way [7]:

$$v_n'\{4\} = -\frac{d_n\{4\}}{(-c_n\{4\})^{3/4}}\,. \qquad (71)$$

We used notation $v_n'\{4\}$ to distinguish estimates for differential flow harmonics v_n' obtained from 4th order cumulants from those which were obtained from 2nd order cumulants and denoted by $v_n'\{2\}$. $v_n'\{4\}$ and $v_n'\{2\}$ are independent estimates for the same differential flow harmonic v_n'. In the case genuine 2-particle nonflow correlations are present in the system $v_n'\{4\}$ is not sensitive to them, while $v_n'\{2\}$ is systematically biased.

E. Example

As an example of differential flow analysis we provide the results for $v_2'(p_t)$ extracted from Therminator dataset. As RPs we select pions and as POIs we select protons. In the first step we estimate reference flow by making use only of particles labeled as RPs, which in this example were pions (relevant equations are (24), (25), (30) and (33)). Results for reference flow estimates are presented on Fig. 7. In the second step we estimate the differential flow of POIs (in this example protons were labeled as POIs) with respect to the reference flow of RPs estimated in the first step. For each p_t bin we evaluate $d_n\{2\}$ and $d_n\{4\}$ from Eq. (64) and (70), respectively, and use equations (65) and (71) to estimate differential flow harmonics. The results for differential flow of protons are presented in Fig. 8.

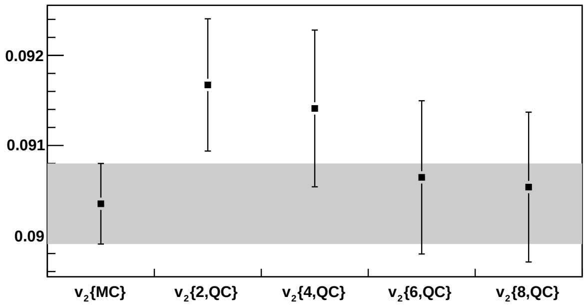

FIG. 7: Reference flow extracted from Therminator dataset and estimated only by making use of particles labeled as RPs, which in this example were pions. Average multiplicity of pions was $\langle M\rangle = 1322$ in a total number of $N = 1876$ events.

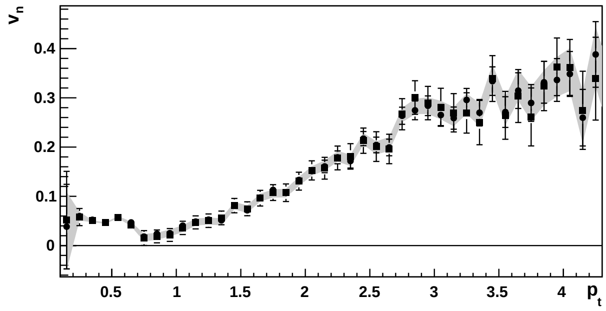

FIG. 8: Differential flow extracted from Therminator dataset for particles labeled as POIs (which in this example were protons—average multiplicity of protons was $\langle M\rangle = 332$ in a total number of $N = 1876$ events.). Differential flow is estimated with respect to the reference flow obtained from particles labeled as RPs (which in this example were pions, see Fig. 7). The grey mesh was obtained after joining the ends of error bars of Monte Carlo estimates for each p_t bin. Closed circles denote 2nd order estimate (Eq. (65) and closed squares denote 4th order estimate (Eq. (71)).

F. Integrated flow

Another useful quantity that is frequently reported as a result of flow measurement is *integrated flow.* In the example presented in previous section we have obtained differential elliptic flow of POIs as a function of transverse momentum, $v_2'(p_t)$. We now define the p_t-integrated flow of POIs in the following way:

$$v_n\{k\} \equiv \frac{\int_0^\infty v_n'\{k\}\frac{dN}{dp_t}dp_t}{\int_0^\infty \frac{dN}{dp_t}dp_t}\,, \qquad (72)$$

where $k = 2,4,6,\ldots$, and $\frac{dN}{dp_t}$ is the yield of POIs in transverse momentum. For the Therminator dataset used in Example IV E the protons were labeled as POIs and their yield in transverse momentum is presented on Fig. 9. The resulting p_t-integated flow of protons calculated by making use of Eq. (72) is presented on Fig. 10.

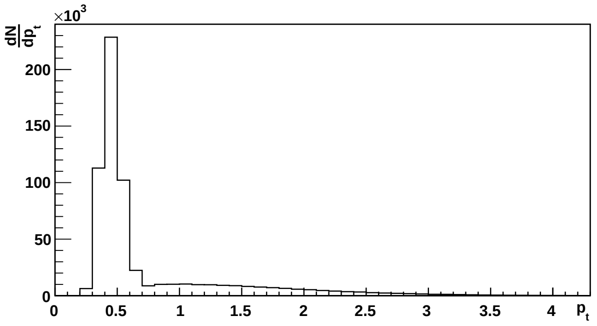

FIG. 9: Transverse momentum proton yield for the Therminator dataset used in Example IV E.

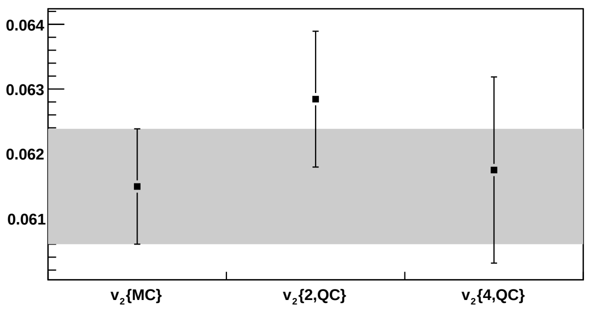

FIG. 10: p_t-integrated flow calculated from Eq. (72) of protons whose differential flow is presented on Fig. 8 and transverse momentum yield on Fig. 9.

V. EPILOGUE

In this paper we have provided a new analytic way to calculate Q-cumulants, advantages of which were outlined in the main part of the paper and illustrated with few examples. When it comes to the usage cases, Q-cumulants will perform best in the system characterized with sizable flow and large multiplicity as it is expected in the heavy-ion collisions. In this regime Q-cumulants are precision method. There are some issues, however, which still has to be resolved analytically and which we now briefly discuss.

A. Open issues

There are three remaining major issues that we couldn't resolve analytically: multiplicity fluctuations, flow fluctuations and low sensitivity for small flow values. In the following lines we discuss briefly their importance.

1. *Multiplicity fluctuations*

When only flow correlations are present in the system multiplicity fluctuations do not yield a systematic bias to the flow estimates from Q-cumulants—it is only the interplay between nonflow correlations and multiplicity fluctuations which accounts for the systematic bias that we couldn't resolve analytically. Both cases are illustrated in Fig. 11. In practice, however, one can always eliminate the systematic bias coming from multiplicity fluctuations by performing a flow analysis in a few very narrow centrality bins. The average flow estimate corresponding to the few merged centrality bins is than obtained straightforwardly by statistically averaging the flow estimates obtained for each narrow centrality bin.

2. *Flow fluctuations*

By using multiparticle azimuthal correlations in flow analysis we are actually estimating the averages of various powers of the flow harmonics (for simplicity sake in this section we suppress the harmonic index and use $v \equiv v_n$ instead):

$$\langle\langle 2\rangle\rangle = \left\langle v^2\right\rangle , \tag{73}$$

$$\langle\langle 4\rangle\rangle = \left\langle v^4\right\rangle . \tag{74}$$

What we are after, however, is $\langle v\rangle$. This means that even in the perfect case scenario, namely when only flow correlations are present in the system, the flow estimates obtained by using multiparticle correlators will be biased due to the statistical flow fluctuations, which are unavoidable. When it comes to the flow estimates from Q-cumulants this systematic bias coming from statistical flow fluctuations can be quantified.

We denote by σ_v^2 the variance of the flow harmonic v,

$$\sigma_v^2 = \left\langle v^2\right\rangle - \langle v\rangle^2 , \tag{75}$$

and with the fairly general assumption that $\sigma_v \ll \langle v\rangle$ and by working up to $2^{\rm nd}$ order in σ_v, we have straight from the formal properties of Taylor expansion obtained the following results:

$$v\{2\} = \langle v\rangle + \frac{1}{2}\frac{\sigma_v^2}{\langle v\rangle} , \tag{76}$$

$$v\{4\} = \langle v\rangle - \frac{1}{2}\frac{\sigma_v^2}{\langle v\rangle} , \tag{77}$$

which are valid *irrespectively* of the details of underlying model of flow fluctuations (the result (77) remains valid also for the estimates from higher order Q-cumulants). The results (76) and (77) are illustrated on Fig. 12 for Gaussian and uniform flow fluctuations.

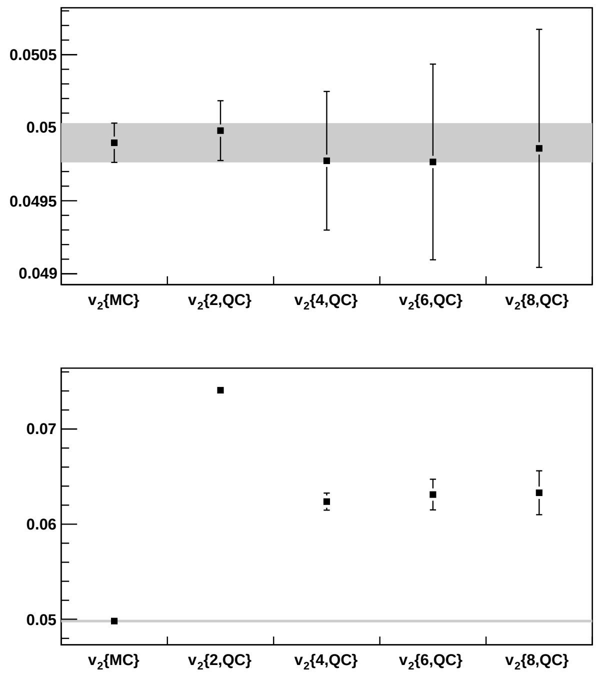

FIG. 11: (*top*) In this example M particles per event were sampled from the azimuthal distribution (1) characterized by harmonic $v_2 = 0.05$. Multiplicity M itself was sampled uniformly event-by-event from the interval $[50, 500]$. In the case only flow correlations are present in the system estimates from Q-cumulants are not biased by multiplicity fluctuations. (*bottom*) In this example M particles per event were sampled from the azimuthal distribution (1) characterized by harmonic $v_2 = 0.05$ and each particle was taken twice for the analysis in order to simulate strong 2-particle nonflow correlations. Multiplicity M itself was sampled uniformly event-by-event from the interval $[25, 250]$, so that resulting multiplicity fluctuates as in the top plot in interval $[50, 500]$. Due to the interplay between nonflow correlations and multiplicity fluctuations also the estimates from higher order cumulants are systematically biased. (In both examples total number of events was $N = 10^5$.)

Since the systematic bias coming from flow fluctuations is quantified as $\pm\frac{1}{2}\frac{\sigma_v^2}{\langle v\rangle}$, it is clear that in the case when flow signal is sizable this bias is negligible. In another limit when flow signal is small the effects of flow fluctuations are best suppressed in estimating the true value of flow harmonic v by making use of the arithmetic average of $v\{2\}$ and $v\{4\}$, because the contribution $\pm\frac{1}{2}\frac{\sigma_v^2}{\langle v\rangle}$ drops out from this average.

3. *Low sensitivity for small values of flow and multiplicity*

Methods for flow analysis based on multiparticle correlations are not sensitive in the regime of very small

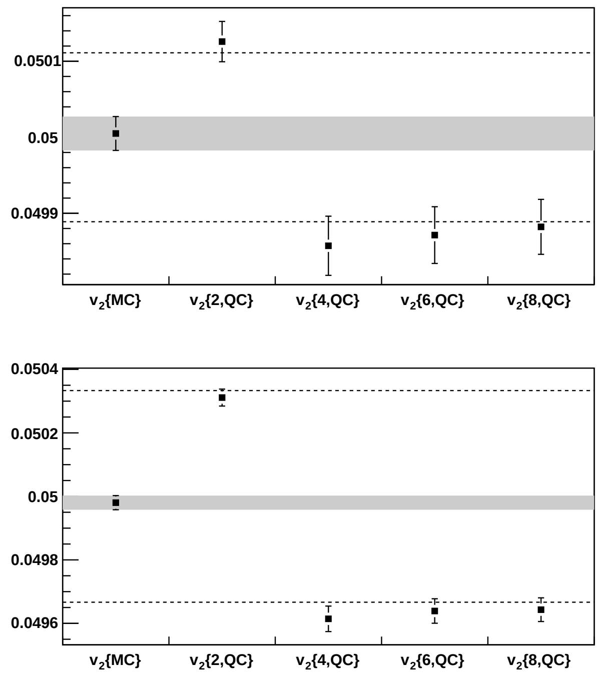

FIG. 12: (*top*) In this example 500 particles per event were sampled from the azimuthal distribution (1) characterized by harmonic v_2, where harmonic v_2 itself was sampled event-by-event from the Gaussian distribution characterized with mean $\langle v\rangle = 0.05$ and $\sigma_v = 1/300$. The dashed lines indicate theoretical results obtained from the Eqs. (76) and (77) for these values of $\langle v\rangle$ and σ_v. Total number of events was $N = 2\times 10^6$. (*bottom*) In this example 500 particles per event were sampled from the azimuthal distribution (1) characterized by harmonic v_2, where harmonic v_2 itself was event-by-event sampled uniformly from the interval $[0.04, 0.06]$. For uniform fluctuations we have that $\langle v\rangle = \frac{v_{\rm max}+v_{\rm min}}{2} = 0.05$ and $\sigma_v = \frac{1}{2\sqrt{3}}(v_{\rm max} - v_{\rm min}) \simeq 0.00577$. The dashed lines indicate theoretical results obtained from Eqs. (76) and (77) for these values of $\langle v\rangle$ and σ_v. Total number of events was $N = 2\times 10^6$.

values of flow and multiplicity. This can be understood from the way the statistical error of flow estimates scales with number of events N, flow signal v and multiplicity M. For instance, for the statistical error of 2$^{\rm nd}$ order estimate we have:

$$s(v, M, N) \sim \frac{1}{\sqrt{N}vM}\,, \tag{78}$$

which means that for the very small values of v and M statistical error s is huge. In practice this means that methods based on multiparticle correlations, in particular Q-cumulants, *cannot* be used straightforwardly to estimate flow harmonics in pp datasets. However, in another regime, namely in heavy-ion datasets where v is sizable and M is large, Q-cumulants can be regarded as a precision method.

APPENDIX A: DERIVATION OF ANALYTIC EXPRESSION FOR $\langle 4\rangle$

The key result in obtaining the analytic result for 4$^{\rm th}$ order Q-cumulant outlined in Eq. (33) was the analytic expression for average single-event multiparticle correlation $\langle 4\rangle$ presented in Eq. (32)—in this Appendix we provide the step-by-step derivation of this result. For completeness sake we start by outlining the definitions of all quantities needed in the derivation:

$$Q_n \equiv \sum_{i=1}^{M} e^{in\phi_i}\,, \tag{A1}$$

$$\langle 2\rangle \equiv \langle 2\rangle_{n|n} \equiv \frac{1}{\binom{M}{2}2!}\sum_{\substack{i,j=1\\(i\neq j)}}^{M} e^{in(\phi_i-\phi_j)}\,, \tag{A2}$$

$$\langle 2\rangle_{2n|2n} \equiv \frac{1}{\binom{M}{2}2!}\sum_{\substack{i,j=1\\(i\neq j)}}^{M} e^{i2n(\phi_i-\phi_j)}\,, \tag{A3}$$

$$\langle 3\rangle_{2n|n,n} \equiv \frac{1}{\binom{M}{3}3!}\sum_{\substack{i,j,k=1\\(i\neq j\neq k)}}^{M} e^{in(2\phi_i-\phi_j-\phi_k)}\,, \tag{A4}$$

$$\langle 3\rangle_{n,n|2n} \equiv \langle 3\rangle_{2n|n,n}^{*}\,, \tag{A5}$$

$$\langle 4\rangle \equiv \langle 4\rangle_{n,n|n,n} \equiv \frac{1}{\binom{M}{4}4!}\sum_{\substack{i,j,k,l=1\\(i\neq j\neq k\neq l)}}^{M} e^{in(\phi_i+\phi_j-\phi_k-\phi_l)}, \tag{A6}$$

where ϕ_i denotes the azimuthal angle of i-th particle measured in laboratory frame and M is multiplicity of event. In the decomposition of $|Q_n|^4$ we have the following multiparticle correlations with corresponding combinatorial coefficients:

$$\begin{aligned}
4-\text{particle} &: \langle 4\rangle_{n,n|n,n}\cdot\binom{M}{4}4!\,,\\
3-\text{particle} &: \langle 3\rangle_{2n|n,n}\cdot M\binom{M-1}{2}2!\,,\\
&\ \langle 3\rangle_{n,n|2n}\cdot M\binom{M-1}{2}2!\,,\\
&\ \langle 2\rangle_{n|n}\cdot M(M-1)2!(M-2)2!\,,\\
2-\text{particle} &: \langle 2\rangle_{n|n}\cdot M(M-1)2!2!\,,\\
&\ \langle 2\rangle_{2n|2n}\cdot M(M-1)\,,\\
&\ 1\cdot\binom{M}{2}2!2!\,,\\
1-\text{particle} &: 1\cdot M\,.
\end{aligned} \tag{A7}$$

Written explicitly (after grouping some terms),

$$\begin{aligned}
|Q_n|^4 &= \langle 4\rangle_{n,n|n,n}\cdot\binom{M}{4}4!\\
&+ \left[\langle 3\rangle_{2n|n,n}+\langle 3\rangle_{n,n|2n}\right]\cdot M\binom{M-1}{2}2!\\
&+ \langle 2\rangle_{n|n}\cdot\left[(M(M-1)^2 2!2!\right]\\
&+ \langle 2\rangle_{2n|2n}\cdot M(M-1)\\
&+ 1\cdot\left[\binom{M}{2}2!2!+M\right]\,.
\end{aligned} \tag{A8}$$

The 2-particle correlations $\langle 2\rangle_{n|n}$ was already expressed in terms of Q-vector evaluated in harmonic n, see Eq. (29), while the analogous expression for $\langle 2\rangle_{2n|2n}$ is trivially

$$\langle 2\rangle_{2n|2n} = \frac{|Q_{2n}|^2-M}{M(M-1)}\,. \tag{A9}$$

What remains to be done here is to express $\langle 3\rangle_{2n|n,n}$ and $\langle 3\rangle_{n,n|2n}$ analytically in terms of Q-vectors. In order to accomplish this goal we must decompose the expressions which consist of Q-vectors evaluated in *different* harmonics. In particular for $\langle 3\rangle_{2n|n,n}$ and $\langle 3\rangle_{n,n|2n}$ we have to decompose $Q_{2n}Q_n^*Q_n^*$ and $Q_nQ_nQ_{2n}^*$, respectively. It follows

$$\begin{aligned}
Q_{2n}Q_n^*Q_n^* &= \langle 3\rangle_{2n|n,n}\cdot M\binom{M}{2}2!+\langle 2\rangle_{n|n}\cdot M(M-1)2!\\
&+ \langle 2\rangle_{2n|2n}\cdot M(M-1)+1\cdot M\,.
\end{aligned} \tag{A10}$$

After inserting results for $\langle 2\rangle_{n|n}$ and $\langle 2\rangle_{2n|2n}$ given in Eqs. (29) and (A9), respectively, and solving for $\langle 3\rangle_{2n|n,n}$ we obtain the following result:

$$\langle 3\rangle_{2n|n,n} = \frac{Q_{2n}Q_n^*Q_n^*-2\cdot|Q_n|^2-|Q_{2n}|^2+2M}{M(M-1)(M-2)}\,. \tag{A11}$$

Trivially,

$$\langle 3\rangle_{n,n|2n} = \frac{Q_nQ_nQ_{2n}^*-2\cdot|Q_n|^2-|Q_{2n}|^2+2M}{M(M-1)(M-2)}\,. \tag{A12}$$

It is easy to see that

$$Q_nQ_nQ_{2n}^* = \left(Q_{2n}Q_n^*Q_n^*\right)^*\,, \tag{A13}$$

so that

$$Q_nQ_nQ_{2n}^* + (Q_{2n}Q_n^*Q_n^*)^* = 2\cdot\mathfrak{Re}\left[Q_{2n}Q_n^*Q_n^*\right]\,. \tag{A14}$$

Taking into account this result we arrive at the following equality:

$$\begin{aligned}
\langle 3\rangle_{n,n|2n}+\langle 3\rangle_{2n|n,n} &= 2\,\frac{\mathfrak{Re}\left[Q_{2n}Q_n^*Q_n^*\right]-2\cdot|Q_n|^2}{M(M-1)(M-2)}\\
&- 2\,\frac{|Q_{2n}|^2-2M}{M(M-1)(M-2)}\,.
\end{aligned} \tag{A15}$$

After inserting results (29), (A9) and (A15) into Eq. (A8) and solving the resulting expression for $\langle 4\rangle_{n,n|n,n}$ the result (32) follows.

1. Higher orders

We remark that this derivation can be straightforwardly (if tediously) generalized to obtain the analytic results for any higher order multiparticle azimuthal correlations. As an example we outline the definition and final analytic result for average single-event 6-particle correlation:

$$
\begin{aligned}
\langle 6\rangle \equiv\ & \frac{1}{\binom{M}{6}6!}\sum_{\substack{i,j,k,l,m,n=1\\(i\neq j\neq k\neq l\neq m\neq n)}}^{M} e^{in(\phi_i+\phi_j+\phi_k-\phi_l-\phi_m-\phi_n)}\\
=\ & \frac{|Q_n|^6+9\cdot|Q_{2n}|^2|Q_n|^2-6\cdot\mathfrak{Re}\left[Q_{2n}Q_nQ_n^*Q_n^*Q_n^*\right]}{M(M-1)(M-2)(M-3)(M-4)(M-5)}\\
+\ & 4\,\frac{\mathfrak{Re}\left[Q_{3n}Q_n^*Q_n^*Q_n^*\right]-3\cdot\mathfrak{Re}\left[Q_{3n}Q_{2n}^*Q_n^*\right]}{M(M-1)(M-2)(M-3)(M-4)(M-5)}\\
+\ & 2\,\frac{9(M-4)\cdot\mathfrak{Re}\left[Q_{2n}Q_n^*Q_n^*\right]+2\cdot|Q_{3n}|^2}{M(M-1)(M-2)(M-3)(M-4)(M-5)}\\
-\ & 9\,\frac{|Q_n|^4+|Q_{2n}|^2}{M(M-1)(M-2)(M-3)(M-5)}\\
+\ & 18\,\frac{|Q_n|^2}{M(M-1)(M-2)(M-3)(M-4)}\\
-\ & \frac{6}{(M-1)(M-2)(M-3)}\,.
\end{aligned}
\tag{A16}
$$

Having obtained this results, the 6$^{\rm th}$ order Q-cumulants is than estimated from

$$
c_n\{6\} = \langle\langle 6\rangle\rangle - 9\cdot\langle\langle 2\rangle\rangle\langle\langle 4\rangle\rangle + 12\cdot\langle\langle 2\rangle\rangle^3\,, \tag{A17}
$$

and reference flow harmonic v_n is estimated as

$$
v_n\{6\} = \sqrt[6]{\frac{1}{4}c_n\{6\}}\,. \tag{A18}
$$

APPENDIX B: PARTICLE WEIGHTS

In this Appendix we provide analytic formulas for the average multiparticle correlations when most general particle weights are used. Standard examples for particle weights are w_ϕ-weights which are being used to correct for detector inefficiencies (this technique is applicable only if there is no gap in detector's azimuthal acceptance, see example in B 1 a) and w_{p_t}-weights which are being used to optimize the flow signal and correspondingly to reduce the spread of flow estimates (see example in ??).

In general we denote the particle weight with w and use it only to weight the contributions of the reference particles (RPs) to Q-vector defined in Eq. (4) and to q-vector defined in Eq. (59). We allow particle weight w to be the most general function of RP's azimuth, transverse momentum and rapidity:

$$
w = w(\phi, p_t, y)\,. \tag{B1}
$$

In the most cases of interest, however, particle weight will be expressed in the factorized form,

$$
w = w_\phi(\phi)w_{p_t}(p_t)w_y(y)\,. \tag{B2}
$$

In order to accommodate the usage of particle weights we introduce the following generalized expression for the weighted Q-vector evaluated in harmonic n:

$$
Q_{n,k} \equiv \sum_{i=1}^{M} w_i^k e^{in\phi_i}\,, \tag{B3}
$$

where w_i is a particle weight of the i-th particle labeled as RP and M is the total number of RPs in particular event. In order to evaluate $Q_{n,k}$ for particular event from Eq. (B3) *all* particles labeled as RPs in that event shall be taken.

Next we focus on the phase window of interest and all particles belonging to that window we label as POI. Taking all POIs from the phase window of interest within particular event (m_p in total) we build up the following quantity for that phase window:

$$
p_{n,k} \equiv \sum_{i=1}^{m_p} w_i^k e^{in\psi_i}\,. \tag{B4}
$$

The important thing to note here is that some of the POIs contributing to $p_{n,k}$ could have also been labeled as RPs. Only for those particles non-unit weight w_i should be used in Eq. (B4), while for the particles labeled *only* as POI the unit weight, $w_i = 1$, must be inserted in Eq. (B4). This difference originates from the fact that in our approach only particles labeled as RPs are used for instance to correct for detector inefficiencies with w_ϕ-weights or to optimize the flow signal (reduce spread) with w_{p_t}-weights. For convenience sake for the subset of POIs which consists of all particles from the phase window of interest within particular event and labeled *both* as POI and RP (m_q in total) we introduce

$$
q_{n,k} \equiv \sum_{i=1}^{m_q} w_i^k e^{in\psi_i}\,. \tag{B5}
$$

The fundamental difference between $Q_{n,k}$ on one side and $p_{n,k}$ and $q_{n,k}$ on another is that for a single event there is a single $Q_{n,k}$ characterizing that event and as many $p_{n,k}$'s and $q_{n,k}$'s as there are phase windows (bins) of interest in that event (for example, when estimating $v'_n(p_t)$ we will evaluate for each p_t bin in each event the dedicated $p_{n,k}$'s and $q_{n,k}$'s).

In order to express some of the results bellow in a more elegant way we will use the following definitions for RPs:

$$
S_{p,k} \equiv \left[\sum_{i=1}^{M} w_i^k\right]^p\,, \tag{B6}
$$

$$
\mathcal{M}_{abcd\cdots} \equiv \sum_{\substack{i,j,k,l,\ldots=1\\(i\neq j\neq k\neq l\neq\ldots)}}^{M} w_i^a w_j^b w_k^c w_l^d\cdots\,. \tag{B7}
$$

Since the summations in Eqs. (B6) and (B7) go over all RPs in a particular event we qualify $S_{p,k}$ and $\mathcal{M}_{abcd\cdots}$ as the *event-wise* quantities.

On the other hand for each phase window of interest in each event we select all particles labeled *both* as RP and POI and evaluate the following quantities:

$$s_{p,k} \equiv \left[\sum_{i=1}^{m_q} w_i^k\right]^p , \tag{B8}$$

$$\mathcal{M}'_{abcd\cdots} \equiv \sum_{i=1}^{m_p} \sum_{\substack{j,k,l,\ldots=1 \\ (i\neq j\neq k\neq l\neq\ldots)}}^{M} w_i^a w_j^b w_k^c w_l^d \cdots . \tag{B9}$$

The summation in Eq. (B8) runs over particles labeled *both* as RP and POI in a phase window (bin) of interest in a particular event, the first summation in Eq. (B9) runs over all POIs (some of which might have been also labeled as RPs) in a phase window of interest in a particular event, while the remaining summations in (B9) run over all RPs in a particular event (hence including the ones which are out of the phase window of interest). The values of $s_{p,k}$ and $\mathcal{M}'_{abcd\cdots}$ will be different for different phase window (bins) of interest within particular event, so we qualify them as a *bin-wise* quantities.

Finally, we remark that we will provide formulas which express analytically $\mathcal{M}_{abcd\cdots}$ and $\mathcal{M}'_{abcd\cdots}$ in terms of $S_{p,k}$'s and $s_{p,k}$'s for which to evaluate only a single loop over data is required.

1. Weighted multiparticle azimuthal correlations

The definitions presented in Section II A for multiparticle azimuthal correlations straightforwardly generalize in the case most general particle weights (B1) are being used. In particular, the *weighted* single-event 2- and 4-particle correlations are defined as:

$$\langle 2\rangle \equiv \frac{1}{\mathcal{M}_{11}} \sum_{\substack{i,j=1 \\ (i\neq j)}}^{M} w_i w_j\, e^{in(\phi_i-\phi_j)} , \tag{B10}$$

$$\langle 4\rangle \equiv \frac{1}{\mathcal{M}_{1111}} \sum_{\substack{i,j,k,l=1 \\ (i\neq j\neq k\neq l)}}^{M} w_i w_j w_k w_l\, e^{in(\phi_i+\phi_j-\phi_k-\phi_l)} , \tag{B11}$$

where $\mathcal{M}_{11}$ and $\mathcal{M}_{1111}$ are determined from definition (B7). On the other hand the event weights (10) and (11) now read

$$W_{\langle 2\rangle} \equiv \mathcal{M}_{11} , \tag{B12}$$

$$W_{\langle 4\rangle} \equiv \mathcal{M}_{1111} . \tag{B13}$$

Analogously, for the reduced single-event multiparticle correlations we have more general definitions when particle weights are being used. In particular the definitions (52) and (53) are now generalized into:

$$\langle 2'\rangle \equiv \frac{1}{\mathcal{M}'_{01}} \sum_{i=1}^{m_p} \sum_{\substack{j=1 \\ (i\neq j)}}^{M} w_j\, e^{in(\psi_i-\phi_j)} , \tag{B14}$$

$$\langle 4'\rangle \equiv \frac{1}{\mathcal{M}'_{0111}} \sum_{i=1}^{m_p} \sum_{\substack{j,k,l=1 \\ (i\neq j\neq k\neq l)}}^{M} w_j w_k w_l\, e^{in(\psi_i+\phi_j-\phi_k-\phi_l)} , \tag{B15}$$

where $\mathcal{M}'_{01}$ and $\mathcal{M}'_{0111}$ are determined from definition (B9). Event weights (56) and (57) now read

$$W_{\langle 2'\rangle} \equiv \mathcal{M}'_{01} , \tag{B16}$$

$$W_{\langle 4'\rangle} \equiv \mathcal{M}'_{0111} . \tag{B17}$$

Having introduced all required definitions, we now present our results. The weighted average 2-particle correlations are given analytically by the following equations:

$$\begin{aligned}
\langle 2\rangle &= \frac{|Q_{n,k}|^2 - S_{1,2}}{S_{2,1}-S_{1,2}} , \\
\langle\langle 2\rangle\rangle &= \frac{\sum_{i=1}^{N}(\mathcal{M}_{11})_i\,\langle 2\rangle_i}{\sum_{i=1}^{N}(\mathcal{M}_{11})_i} , \\
\mathcal{M}_{11} &\equiv \sum_{\substack{i,j=1 \\ (i\neq j)}}^{M} w_i w_j \\
&= S_{2,1} - S_{1,2} ,
\end{aligned} \tag{B18}$$

and the weighted average 4-particle correlations are given by:

$$\begin{aligned}
\langle 4\rangle &= \Big[|Q_{n,1}|^4 + |Q_{2n,2}|^2 - 2\cdot \mathfrak{Re}\left[Q_{2n,2}Q^*_{n,1}Q^*_{n,1}\right] \\
&+ 8\cdot \mathfrak{Re}\left[Q_{n,3}Q^*_{n,1}\right] - 4\cdot S_{1,2}\,|Q_{n,1}|^2 \\
&- 6\cdot S_{1,4} - 2\cdot S_{2,2}\Big] / \mathcal{M}_{1111} , \\
\mathcal{M}_{1111} &\equiv \sum_{\substack{i,j,k,l=1 \\ (i\neq j\neq k\neq l)}}^{M} w_i w_j w_k w_l \\
&= S_{4,1} - 6\cdot S_{1,2}S_{2,1} + 8\cdot S_{1,3}S_{1,1} + 3\cdot S_{2,2} \\
&- 6\cdot S_{1,4} , \\
\langle\langle 4\rangle\rangle &= \frac{\sum_{i=1}^{N}(\mathcal{M}_{1111})_i\,\langle 4\rangle_i}{\sum_{i=1}^{N}(\mathcal{M}_{1111})_i} ,
\end{aligned} \tag{B19}$$

where weighted Q-vector, $Q_{n,k}$, was defined in Eq. (B3) and $S_{p,k}$ in Eq. (B6).

When it comes to weighted reduced 2- and 4-particle azimuthal correlations they are given analytically by the following formulas:

$$
\begin{aligned}
\langle 2' \rangle &= \frac{p_{n,0}Q_{n,k}^* - s_{1,1}}{m_p S_{1,1} - s_{1,1}}\,,\\
\langle\langle 2' \rangle\rangle &= \frac{\sum_{i=1}^{N}(\mathcal{M}'_{01})_i \, \langle 2' \rangle_i}{\sum_{i=1}^{N}(\mathcal{M}'_{01})_i}\,,\\
\mathcal{M}'_{01} &\equiv \sum_{i=1}^{m_p}\sum_{\substack{j=1\\(i\neq j)}}^{M} w_j\\
&= m_p S_{1,1} - s_{1,1}\,, \qquad \text{(B20)}
\end{aligned}
$$

and,

$$
\begin{aligned}
\langle 4' \rangle = \Big[& p_{n,0}Q_{n,1}Q_{n,1}^*Q_{n,1}^*\\
- \; & q_{2n,1}Q_{n,1}^*Q_{n,1}^* - p_{n,0}Q_{n,1}Q_{2n,2}^*\\
- \; & 2\cdot S_{1,2}p_{n,0}Q_{n,1}^* - 2\cdot s_{1,1}\,|Q_{n,1}|^2\\
+ \; & 7\cdot q_{n,2}Q_{n,1}^* - Q_{n,1}q_{n,2}^*\\
+ \; & q_{2n,1}Q_{2n,2}^* + 2\cdot p_{n,0}Q_{n,3}^*\\
+ \; & 2\cdot s_{1,1}S_{1,2} - 6\cdot s_{1,3}\Big] / \mathcal{M}'_{0111}\,,
\end{aligned}
$$

$$
\begin{aligned}
\langle\langle 4' \rangle\rangle &= \frac{\sum_{i=1}^{N}(\mathcal{M}'_{0111})_i \, \langle 4' \rangle_i}{\sum_{i=1}^{N}(\mathcal{M}'_{0111})_i}\,,\\
\mathcal{M}'_{0111} &\equiv \sum_{i=1}^{m_p}\sum_{\substack{j,k,l=1\\(i\neq j\neq k\neq l)}}^{M} w_j w_k w_l\\
&= m_p\left[S_{3,1} - 3\cdot S_{1,1}S_{1,2} + 2\cdot S_{1,3}\right]\\
&- 3\cdot[s_{1,1}(S_{2,1}-S_{1,2})+2\cdot(s_{1,3}-s_{1,2}S_{1,1})] \qquad \text{(B21)}
\end{aligned}
$$

We remark that to evaluate all quantities appearing on the right hand sides in analytic expressions (B18–B21) only a single loop over data is required. Next we provide two illustrative examples for the usage of particle weights.

a. *Example for w_ϕ-weights: Correcting for the bias coming from non-uniform acceptance of detector*

In this example we sample particles from the azimuthal distribution (1) characterized with p_t dependent harmonic v_2, namely:

$$
v_2(p_t) = \begin{cases} 0.2\,(p_t/2.0) & p_t < 2.0 \text{ GeV}\,,\\ 0.2 & p_t \geq 2.0 \text{ GeV}\,. \end{cases} \qquad \text{(B22)}
$$

The usage of w_ϕ-weights is illustrated for the detector with two problematic sectors, namely for the detector which accepts only 1/2 of the tracks going in $60^{\rm o} \leq \phi < 100^{\rm o}$ and only 1/3 of the tracks going in $270^{\rm o} \leq \phi < 330^{\rm o}$. All particles going in azimuthal angles out of these two ranges are accepted without any loss.

In the first step we have to perform a dedicated run over data in order to get the histogram of detector's azimuthal acceptance. The resulting acceptance histogram is presented on Fig. 13 (top). In the second step from this histogram the normalized w_ϕ-weights are constructed. If the average number of particles per ϕ-bin is $\langle N\rangle$ and if the number of particles in particular ϕ-bin is N_ϕ for the histogram on Fig. 13 (top), than the normalized w_ϕ-weight for that ϕ-bin is simply

$$
w_\phi \equiv \frac{\langle N\rangle}{N_\phi}\,. \qquad \text{(B23)}
$$

The resulting normalized w_ϕ-weights for the detector in question are presented in Fig. 13 (bottom). From the Eq. (B23) it is clear that if there is a gap in the detector's acceptance than there will be a ϕ-bin with zero entries and the w_ϕ-weight for that bin cannot be constructed—this limits the applicability of the usage of w_ϕ-weights (to a certain extent one can avoid this problem in practice by making binning in ϕ coarser). Having obtained w_ϕ-weights in a dedicated run, in the subsequent runs over data weighted Q-, p- and q-vectors shall be evaluated as defined in Eqs. (B3), (B4) and (B5), respectively. In addition, quantities $S_{p,k}$ and $s_{p,k}$ shall be evaluated according to definitions (B6) and (B8). From these quantities by making use of analytic results presented in Section B 1 all average multiparticle and reduced multiparticle correlations can be obtained for the case w_ϕ-weights are being used. From this point on the procedure to estimate flow

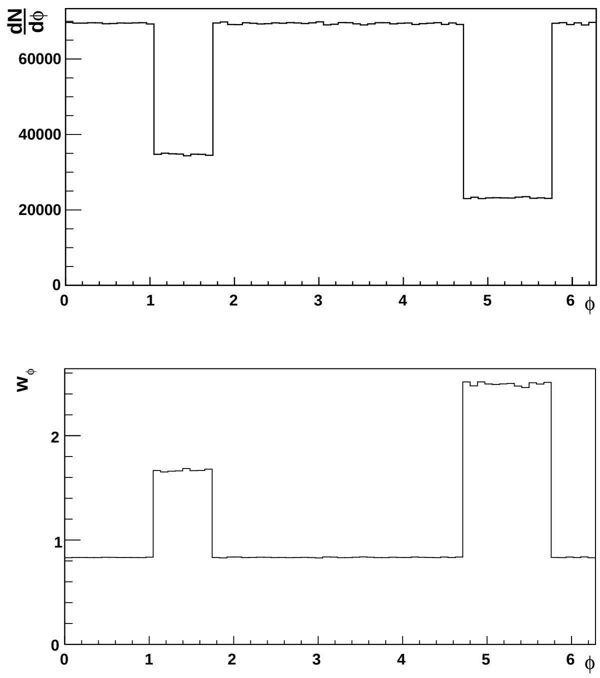

FIG. 13: Azimuthal profile of a detector which accepts 1/2 of the tracks going in $60^{\rm o} \leq \phi < 100^{\rm o}$ and only 1/3 of the tracks going in $270^{\rm o} \leq \phi < 330^{\rm o}$ (*top*). Resulting normalized w_ϕ-weights for this acceptance profile (*bottom*).

harmonics is the same as outlined in the main part of the paper. On Fig. 14 the estimates for reference flow harmonics are presented and on Figs. 15 and 16 the estimates for differential flow harmonics from 2nd and 4th order differential Q-cumulant, respectively, with (closed markers) and without (open markers) using the w_ϕ-weights.

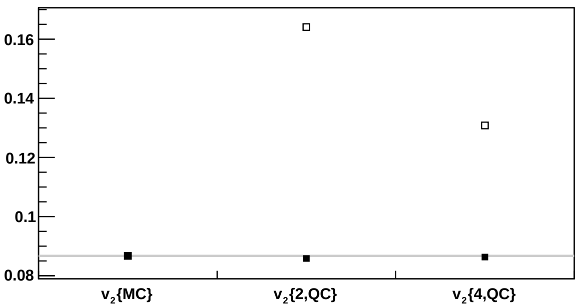

FIG. 14: Estimates for reference flow harmonics for detector whose acceptance histogram is presented on the top plot of Fig. 13. With open markers are estimates obtained without using w_ϕ-weights and with closed markers with using w_ϕ-weights.

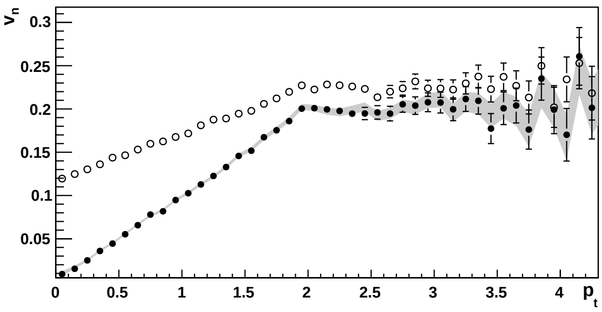

FIG. 15: Estimates for differential flow harmonics from 2nd order differential Q-cumulant for detector whose acceptance histogram is presented on the top plot of Fig. 13. The grey mesh was obtained after joining the ends of error bars of Monte Carlo estimates for each p_t bin. With open markers are estimates obtained without using w_ϕ-weights and with closed markers with using w_ϕ-weights.

APPENDIX C: NON-UNIFORM ACCEPTANCE

In this Appendix we outline the general procedure how the bias coming from non-uniform acceptance of the detector can be quantified and removed from the flow estimates. Claim: In order to use this procedure one run over data is enough and this procedure can be used for all types of non-uniform acceptance. In particular, it can

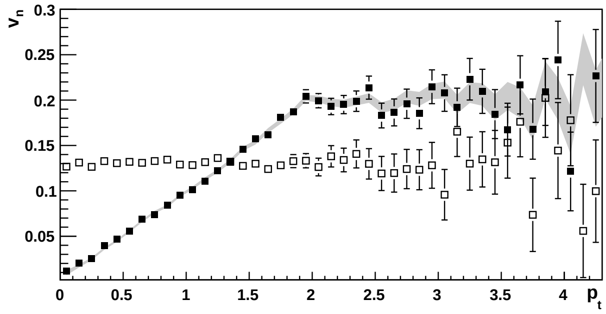

FIG. 16: Estimates for differential flow harmonics from 4th order differential Q-cumulant for detector whose acceptance histogram is presented on the top plot of Fig. 13. The grey mesh was obtained after joining the ends of error bars of Monte Carlo estimates for each p_t bin. With open markers are estimates obtained without using w_ϕ-weights and with closed markers with using w_ϕ-weights.

be used also for the detectors with partial azimuthal coverage (for instance for the PHENIX detector) for which the usage of w_ϕ-weights presented in example in B 1 a is not possible.

1. Generalized Q-cumulants

For the case of detectors with non-uniform acceptance the average multiparticle correlations are strongly biased. On the other hand the cumulants isolate genuine physical correlation between the particles which should not be spoiled at all by any inefficiencies in the detector. This suggests that in building cumulants from multiparticle correlations we have so far omitted terms which vanish for the detectors with uniform acceptance. For the detectors with non-uniform acceptance these terms are crucial and must be called back, simply because they will counter balance the bias on multiparticle correlations, so that cumulants remain unbiased.

The generalized cumulants to be used for detectors with non-uniform acceptance were given in [5, 6, 7]. In [5] the formulas were provided which determine generalized cumulants to all orders, but without any specific connection to azimuthally sensitive observables. In [6, 7] this general formalism has been applied for the azimuthally sensitive obervables yielding the results in which cumulants were composed both from isotropic and anisotropic terms. However, the authors of [6, 7] were pursuing the usage of formalism of generating functions and by making a suitable projections along the fixed, equally spaced directions in the laboratory frame they were able to average out all anisotropic terms which appear in the generalized cumulants. In what follows instead we show that all these anisotropic terms can be also expressed analytically in terms of Q-vectors, which will serve as a basis for explicitly quantifying the bias to flow estimates coming from the non-uniform acceptance of the detector.

a. Reference flow

We start by outlining the generalized Q-cumulants which are used to estimate unbiasedly the reference flow harmonics for any type of detector's acceptance. In Eq. (19) the $2^{\rm nd}$ order Q-cumulant for detectors with uniform acceptance was defined—its generalized version reads:

$$c_n\{2\} = \langle\langle 2\rangle\rangle - \langle\langle\cos n\phi_1\rangle\rangle^2 - \langle\langle\sin n\phi_1\rangle\rangle^2 \,. \tag{C1}$$

The terms in red counter balance the bias to 2-particle correlation due to non-uniform acceptance, so that $c_n\{2\}$ remains unbiased. These new anisotropic terms can be trivially expressed in terms of real and imaginary parts of Q-vector (4):

$$\langle\langle\cos n\phi_1\rangle\rangle = \frac{\sum_{i=1}^N \left(\mathfrak{Re}\left[Q_n\right]\right)_i}{\sum_{i=1}^N M_i}\,, \tag{C2}$$

$$\langle\langle\sin n\phi_1\rangle\rangle = \frac{\sum_{i=1}^N \left(\mathfrak{Im}\left[Q_n\right]\right)_i}{\sum_{i=1}^N M_i}\,, \tag{C3}$$

and average 2-particle correlation $\langle\langle 2\rangle\rangle$ was expressed in terms of Q-vectors in Section III. Note that terms in red in Eq. (C1) explicitly *quantify* the bias coming from non-uniform acceptance to average 2-particle correlation. When particle weights are used the average 2-particle correlation $\langle\langle 2\rangle\rangle$ is determined from Eqs. (B18), while Eqs. (C2) and (C3) generalize into:

$$\langle\langle\cos n\phi_1\rangle\rangle = \frac{\sum_{i=1}^N \left(\mathfrak{Re}\left[Q_{n,1}\right]\right)_i}{\sum_{i=1}^N (S_{1,1})_i}\,, \tag{C4}$$

$$\langle\langle\sin n\phi_1\rangle\rangle = \frac{\sum_{i=1}^N \left(\mathfrak{Im}\left[Q_{n,1}\right]\right)_i}{\sum_{i=1}^N (S_{1,1})_i}\,, \tag{C5}$$

where $Q_{n,1}$ can be determined from definition of weighted Q-vector (B3) and $S_{1,1}$ from definition (B6).

The $4^{\rm th}$ order Q-cumulant to be used to estimate flow harmonics for the case of detectors with uniform acceptance was defined in Eq. (22). The generalized $4^{\rm th}$ order Q-cumulant to be used for any type of detector acceptance reads:

$$\begin{aligned} c_n\{4\} &= \langle\langle 4\rangle\rangle - 2\cdot\langle\langle 2\rangle\rangle^2 \\ &- 4\cdot\langle\langle\cos n\phi_1\rangle\rangle\,\langle\langle\cos n(\phi_1-\phi_2-\phi_3)\rangle\rangle \\ &+ 4\cdot\langle\langle\sin n\phi_1\rangle\rangle\,\langle\langle\sin n(\phi_1-\phi_2-\phi_3)\rangle\rangle \\ &- \langle\langle\cos n(\phi_1+\phi_2)\rangle\rangle^2 - \langle\langle\sin n(\phi_1+\phi_2)\rangle\rangle^2 \\ &+ 4\cdot\langle\langle\cos n(\phi_1+\phi_2)\rangle\rangle \\ &\times \left[\langle\langle\cos n\phi_1\rangle\rangle^2 - \langle\langle\sin n\phi_1\rangle\rangle^2\right] \\ &+ 8\cdot\langle\langle\sin n(\phi_1+\phi_2)\rangle\rangle\,\langle\langle\sin n\phi_1\rangle\rangle\,\langle\langle\cos n\phi_1\rangle\rangle \\ &+ 8\cdot\langle\langle\cos n(\phi_1-\phi_2)\rangle\rangle \\ &\times \left[\langle\langle\cos n\phi_1\rangle\rangle^2 + \langle\langle\sin n\phi_1\rangle\rangle^2\right] \\ &- 6\cdot\left[\langle\langle\cos n\phi_1\rangle\rangle^2 + \langle\langle\sin n\phi_1\rangle\rangle^2\right]^2 . \end{aligned} \tag{C6}$$

The terms in red in Eq. (C6) counter balance the bias coming from non-uniform acceptance so that $c_n\{4\}$ is unbiased. The terms in red in Eq. (C6) explicitly *quantify* the bias coming from non-uniform acceptance. It was explained in Section III how to express $\langle\langle 4\rangle\rangle$ and $\langle\langle 2\rangle\rangle$ in terms of Q-vectors. The new anisotropic terms in Eq. (C6) can also be expressed analytically in terms of Q-vectors. In particular, the 2-particle anisotropic terms read:

$$\langle\langle\cos n(\phi_1+\phi_2)\rangle\rangle = \frac{\sum_{i=1}^N \left(\mathfrak{Re}\left[Q_nQ_n-Q_{2n}\right]\right)_i}{\sum_{i=1}^N M_i(M_i-1)}, \tag{C7}$$

$$\langle\langle\sin n(\phi_1+\phi_2)\rangle\rangle = \frac{\sum_{i=1}^N \left(\mathfrak{Im}\left[Q_nQ_n-Q_{2n}\right]\right)_i}{\sum_{i=1}^N M_i(M_i-1)}, \tag{C8}$$

while for 3-particle anisotropic terms the following results apply:

$$\begin{aligned} \langle\langle\cos n(\phi_1-\phi_2-\phi_3)\rangle\rangle &= \left\{\sum_{i=1}^N \left(\mathfrak{Re}\left[Q_nQ_n^*Q_n^* - Q_nQ_{2n}^*\right]\right.\right. \\ &\left.\left. -2(M-1)\mathfrak{Re}\left[Q_n^*\right]\right)_i\right\} \Big/ \sum_{i=1}^N M_i(M_i-1)(M_i-2)\,, \end{aligned} \tag{C9}$$

$$\begin{aligned} \langle\langle\sin n(\phi_1-\phi_2-\phi_3)\rangle\rangle &= \left\{\sum_{i=1}^N \left(\mathfrak{Im}\left[Q_nQ_n^*Q_n^* - Q_nQ_{2n}^*\right]\right.\right. \\ &\left.\left. -2(M-1)\mathfrak{Im}\left[Q_n^*\right]\right)_i\right\} \Big/ \sum_{i=1}^N M_i(M_i-1)(M_i-2)\,. \end{aligned} \tag{C10}$$

When particle weights are used the average 2-particle correlation $\langle\langle 2\rangle\rangle$ is determined from Eqs. (B18), the average 4-particle correlation $\langle\langle 4\rangle\rangle$ is determined from Eqs.

(B19), the Eqs. (C7) and (C8) generalize into:

$$\begin{aligned}
\langle\langle\cos n(\phi_1+\phi_2)\rangle\rangle &= \frac{\sum_{i=1}^{N}\left(\mathfrak{Re}\left[Q_{n,1}Q_{n,1}-Q_{2n,2}\right]\right)_i}{\sum_{i=1}^{N}(\mathcal{M}_{11})_i}\,,\\
\langle\langle\sin n(\phi_1+\phi_2)\rangle\rangle &= \frac{\sum_{i=1}^{N}\left(\mathfrak{Im}\left[Q_{n,1}Q_{n,1}-Q_{2n,2}\right]\right)_i}{\sum_{i=1}^{N}(\mathcal{M}_{11})_i}\,,\\
\mathcal{M}_{11} &\equiv \sum_{\substack{i,j=1\\(i\neq j)}}^{M} w_i w_j = S_{2,1}-S_{1,2}\,, \qquad \text{(C11)}
\end{aligned}$$

and the Eqs. (C9) and (C10) generalize into

$$\begin{aligned}
\langle\langle\cos n(\phi_1-\phi_2-\phi_3)\rangle\rangle &= \Big\{\sum_{i=1}^{N}\big(\mathfrak{Re}\big[Q_{n,1}Q_{n,1}^{*}Q_{n,1}^{*}\\
&-Q_{n,1}Q_{2n,2}^{*}-2\cdot S_{1,2}Q_{n,1}^{*}+2\cdot Q_{n,3}^{*}\big]\big)_i\Big\}\Big/\sum_{i=1}^{N}(\mathcal{M}_{111})_i\,,\\
\langle\langle\sin n(\phi_1-\phi_2-\phi_3)\rangle\rangle &= \Big\{\sum_{i=1}^{N}\big(\mathfrak{Im}\big[Q_{n,1}Q_{n,1}^{*}Q_{n,1}^{*}\\
&-Q_{n,1}Q_{2n,2}^{*}-2\cdot S_{1,2}Q_{n,1}^{*}+2\cdot Q_{n,3}^{*}\big]\big)_i\Big\}\Big/\sum_{i=1}^{N}(\mathcal{M}_{111})_i\,,\\
\mathcal{M}_{111} &\equiv \sum_{\substack{i,j,k=1\\(i\neq j\neq k)}}^{M} w_i w_j w_k = S_{3,1}-3\cdot S_{1,2}S_{1,1}+2\cdot S_{1,3}\,. \qquad \text{(C12)}
\end{aligned}$$

Next we provide the formulas for generalized differential Q-cumulants.

b. Differential flow

The generalized $2^{\rm nd}$ order differential Q-cumulant reads

$$\begin{aligned}
d_n\{2\} &= \langle\langle 2'\rangle\rangle - \langle\langle\cos n\psi_1\rangle\rangle\,\langle\langle\cos n\phi_2\rangle\rangle\\
&- \langle\langle\sin n\psi_1\rangle\rangle\,\langle\langle\sin n\phi_2\rangle\rangle\,. \qquad \text{(C13)}
\end{aligned}$$

The terms in red counter balance the bias to reduced 2-particle correlation $\langle\langle 2'\rangle\rangle$ coming from the detector's non-uniform acceptance, so that $d_n\{2\}$ remains unbiased. Expressions for $\langle\langle\cos n\phi_1\rangle\rangle$ and $\langle\langle\sin n\phi_1\rangle\rangle$ were already given in Eqs. (C2) and (C3), respectively (when particle weights are being used in Eqs. (C4) and (C5), respectively). Analogously we derive the following results,

$$\begin{aligned}
\langle\langle\cos n\psi_1\rangle\rangle &= \frac{\sum_{i=1}^{N}\left(\mathfrak{Re}\left[p_n\right]\right)_i}{\sum_{i=1}^{N}(m_p)_i}\,, \qquad \text{(C14)}\\
\langle\langle\sin n\psi_1\rangle\rangle &= \frac{\sum_{i=1}^{N}\left(\mathfrak{Im}\left[p_n\right]\right)_i}{\sum_{i=1}^{N}(m_p)_i}\,, \qquad \text{(C15)}
\end{aligned}$$

where p_n and m_p were defined in Section IV. The Eqs. (C14) and (C15) remain unchanged when particle weights are being used.

The generalized $4^{\rm th}$ order differential Q-cumulant reads:

$$\begin{aligned}
d_n\{4\} &= \langle\langle 4'\rangle\rangle - 2\cdot\langle\langle 2'\rangle\rangle\,\langle\langle 2\rangle\rangle \qquad \text{(C16)}\\
&- \langle\langle\cos n\psi_1\rangle\rangle\,\langle\langle\cos n(\phi_1-\phi_2-\phi_3)\rangle\rangle\\
&+ \langle\langle\sin n\psi_1\rangle\rangle\,\langle\langle\sin n(\phi_1-\phi_2-\phi_3)\rangle\rangle\\
&- \langle\langle\cos n\phi_1\rangle\rangle\,\langle\langle\cos n(\psi_1-\phi_2-\phi_3)\rangle\rangle\\
&+ \langle\langle\sin n\phi_1\rangle\rangle\,\langle\langle\sin n(\psi_1-\phi_2-\phi_3)\rangle\rangle\\
&- 2\cdot\langle\langle\cos n\phi_1\rangle\rangle\,\langle\langle\cos n(\psi_1+\phi_2-\phi_3)\rangle\rangle\\
&- 2\cdot\langle\langle\sin n\phi_1\rangle\rangle\,\langle\langle\sin n(\psi_1+\phi_2-\phi_3)\rangle\rangle\\
&- \langle\langle\cos n(\psi_1+\phi_2)\rangle\rangle\,\langle\langle\cos n(\phi_1+\phi_2)\rangle\rangle\\
&- \langle\langle\sin n(\psi_1+\phi_2)\rangle\rangle\,\langle\langle\sin n(\phi_1+\phi_2)\rangle\rangle\\
&+ 2\cdot\langle\langle\cos n(\phi_1+\phi_2)\rangle\rangle\\
&\times \left[\langle\langle\cos n\psi_1\rangle\rangle\,\langle\langle\cos n\phi_1\rangle\rangle-\langle\langle\sin n\psi_1\rangle\rangle\,\langle\langle\sin n\phi_1\rangle\rangle\right]\\
&+ 2\cdot\langle\langle\sin n(\phi_1+\phi_2)\rangle\rangle\\
&\times \left[\langle\langle\cos n\psi_1\rangle\rangle\,\langle\langle\sin n\phi_1\rangle\rangle+\langle\langle\sin n\psi_1\rangle\rangle\,\langle\langle\cos n\phi_1\rangle\rangle\right]\\
&+ 4\cdot\langle\langle\cos n(\phi_1-\phi_2)\rangle\rangle\\
&\times \left[\langle\langle\cos n\psi_1\rangle\rangle\,\langle\langle\cos n\phi_1\rangle\rangle+\langle\langle\sin n\psi_1\rangle\rangle\,\langle\langle\sin n\phi_1\rangle\rangle\right]\\
&+ 2\cdot\langle\langle\cos n(\psi_1+\phi_2)\rangle\rangle\\
&\times \left[\langle\langle\cos n\phi_1\rangle\rangle^2-\langle\langle\sin n\phi_1\rangle\rangle^2\right]\\
&+ 4\cdot\langle\langle\sin n(\psi_1+\phi_2)\rangle\rangle\,\langle\langle\cos n\phi_1\rangle\rangle\,\langle\langle\sin n\phi_1\rangle\rangle\\
&+ 4\cdot\langle\langle\cos n(\psi_1-\phi_2)\rangle\rangle\left[\langle\langle\cos n\phi_1\rangle\rangle^2+\langle\langle\sin n\phi_1\rangle\rangle^2\right]\\
&- 6\cdot\left[\langle\langle\cos n\phi_1\rangle\rangle^2-\langle\langle\sin n\phi_1\rangle\rangle^2\right]\\
&\times \left[\langle\langle\cos n\psi_1\rangle\rangle\,\langle\langle\cos n\phi_1\rangle\rangle-\langle\langle\sin n\psi_1\rangle\rangle\,\langle\langle\sin n\phi_1\rangle\rangle\right]\\
&- 12\cdot\langle\langle\cos n\phi_1\rangle\rangle\,\langle\langle\sin n\phi_1\rangle\rangle\\
&\times \left[\langle\langle\sin n\psi_1\rangle\rangle\,\langle\langle\cos n\phi_1\rangle\rangle+\langle\langle\cos n\psi_1\rangle\rangle\,\langle\langle\sin n\phi_1\rangle\rangle\right].
\end{aligned}$$

The terms in red in Eq. (C16) counter balance the bias coming from non-uniform acceptance, so that $d_n\{4\}$ is unbiased. In this equation some new anisotropic terms appear which have to be expressed analytically in terms of Q-, p- and q-vectors. In particular we have obtained the following results for 2-particle anisotropic terms:

$$\begin{aligned}
\langle\langle\cos n(\psi_1+\phi_2)\rangle\rangle &= \frac{\sum_{i=1}^{N}(\mathfrak{Re}\left[p_nQ_n-q_{2n}\right])_i}{\sum_{i=1}^{N}(m_pM-m_q)_i}\,,\\
\langle\langle\sin n(\psi_1+\phi_2)\rangle\rangle &= \frac{\sum_{i=1}^{N}(\mathfrak{Im}\left[p_nQ_n-q_{2n}\right])_i}{\sum_{i=1}^{N}(m_pM-m_q)_i}\,, \qquad \text{(C17)}
\end{aligned}$$

and the following results for 3-particle anisotropic terms:

$$
\begin{aligned}
\langle\langle\cos n(\psi_1+\phi_2-\phi_3)\rangle\rangle &= \left\{\sum_{i=1}^{N}\left(\mathfrak{Re}\left[p_n\left(|Q_n|^2-M\right)\right]\right.\right.\\
&\left.\left.-\mathfrak{Re}\left[q_{2n}Q_n^*+m_qQ_n-2q_n\right]\right)_i\right\}/\sum_{i=1}^{N}\left[(m_pM-2m_q)(M-1)\right]_i\,,\\
\langle\langle\sin n(\psi_1+\phi_2-\phi_3)\rangle\rangle &= \left\{\sum_{i=1}^{N}\left(\mathfrak{Im}\left[p_n\left(|Q_n|^2-M\right)\right]\right.\right.\\
&\left.\left.-\mathfrak{Im}\left[q_{2n}Q_n^*+m_qQ_n-2q_n\right]\right)_i\right\}/\sum_{i=1}^{N}\left[(m_pM-2m_q)(M-1)\right]_i
\end{aligned}
\tag{C18}
$$

and for the additional 3-particle anisotropic terms:

$$
\begin{aligned}
\langle\langle\cos n(\psi_1-\phi_2-\phi_3)\rangle\rangle &= \left\{\sum_{i=1}^{N}\left(\mathfrak{Re}\left[p_nQ_n^*Q_n^*-p_nQ_{2n}^*\right]\right.\right.\\
&\left.\left.-\mathfrak{Re}\left[2m_qQ_n^*-2q_n^*\right]\right)_i\right\}/\sum_{i=1}^{N}\left[(m_pM-2m_q)(M-1)\right]_i\,,\\
\langle\langle\sin n(\psi_1-\phi_2-\phi_3)\rangle\rangle &= \left\{\sum_{i=1}^{N}\left(\mathfrak{Im}\left[p_nQ_n^*Q_n^*-p_nQ_{2n}^*\right]\right.\right.\\
&\left.\left.-\mathfrak{Im}\left[2m_qQ_n^*-2q_n^*\right]\right)_i\right\}/\sum_{i=1}^{N}\left[(m_pM-2m_q)(M-1)\right]_i\,.
\end{aligned}
\tag{C19}
$$

When particle weights are being used Eqs. (C17) generalize into:

$$
\begin{aligned}
\langle\langle\cos n(\psi_1+\phi_2)\rangle\rangle &= \frac{\sum_{i=1}^{N}\left(\mathfrak{Re}\left[p_nQ_{n,k}-q_{2n,k}\right]\right)_i}{\sum_{i=1}^{N}\left(m_pS_{1,1}-s_{1,1}\right)_i}\,,\\
\langle\langle\sin n(\psi_1+\phi_2)\rangle\rangle &= \frac{\sum_{i=1}^{N}\left(\mathfrak{Im}\left[p_nQ_{n,k}-q_{2n,k}\right]\right)_i}{\sum_{i=1}^{N}\left(m_pS_{1,1}-s_{1,1}\right)_i}\,,
\end{aligned}
\tag{C20}
$$

Eqs. (C18) generalize into:

$$
\begin{aligned}
&\langle\langle\cos n(\psi_1+\phi_2-\phi_3)\rangle\rangle=\left\{\sum_{i=1}^{N}\left(\mathfrak{Re}\left[p_n\left(|Q_{n,1}|^2-S_{1,2}\right)\right]\right.\right.\\
&\left.\left.-\mathfrak{Re}\left[q_{2n,1}Q_{n,1}^*+s_{1,1}Q_{n,1}-2q_{n,2}\right]\right)_i\right\}/\\
&\left\{\sum_{i=1}^{N}\left(m_p(S_{2,1}-S_{1,2})-2\cdot(s_{1,1}S_{1,1}-s_{1,2})\right)_i\right\},\\
&\langle\langle\sin n(\psi_1+\phi_2-\phi_3)\rangle\rangle=\left\{\sum_{i=1}^{N}\left(\mathfrak{Im}\left[p_n\left(|Q_{n,1}|^2-S_{1,2}\right)\right]\right.\right.\\
&\left.\left.-\mathfrak{Im}\left[q_{2n,1}Q_{n,1}^*+s_{1,1}Q_{n,1}-2q_{n,2}\right]\right)_i\right\}/\\
&\left\{\sum_{i=1}^{N}\left(m_p(S_{2,1}-S_{1,2})-2\cdot(s_{1,1}S_{1,1}-s_{1,2})\right)_i\right\},
\end{aligned}
\tag{C21}
$$

and finally, Eqs. (C19) generalize into:

$$
\begin{aligned}
&\langle\langle\cos n(\psi_1-\phi_2-\phi_3)\rangle\rangle=\left\{\sum_{i=1}^{N}\left(\mathfrak{Re}\left[p_n\left(Q_{n,1}^*Q_{n,1}^*-Q_{2n,2}^*\right)\right]\right.\right.\\
&\left.\left.-2\cdot\mathfrak{Re}\left[s_{1,1}Q_{n,1}^*-q_{n,2}^*\right]\right)_i\right\}/\\
&\left\{\sum_{i=1}^{N}\left(m_p(S_{2,1}-S_{1,2})-2\cdot(s_{1,1}S_{1,1}-s_{1,2})\right)_i\right\},\\
&\langle\langle\sin n(\psi_1-\phi_2-\phi_3)\rangle\rangle=\left\{\sum_{i=1}^{N}\left(\mathfrak{Im}\left[p_n\left(Q_{n,1}^*Q_{n,1}^*-Q_{2n,2}^*\right)\right]\right.\right.\\
&\left.\left.-2\cdot\mathfrak{Im}\left[s_{1,1}Q_{n,1}^*-q_{n,2}^*\right]\right)_i\right\}/\\
&\left\{\sum_{i=1}^{N}\left(m_p(S_{2,1}-S_{1,2})-2\cdot(s_{1,1}S_{1,1}-s_{1,2})\right)_i\right\}.
\end{aligned}
\tag{C22}
$$

APPENDIX D: STATISTICAL ERRORS

In this appendix we outline the procedure we use to report the statistical errors of the reference and differential flow estimates from Q-cumulants. For simplicity sake in all expressions bellow we keep only the terms relevant for detectors with uniform acceptance—more general expressions for detectors with non-uniform acceptance can be derived straightforwardly.

1. Some general results

Consider the random observable x sampled from some probability density function (p.d.f.) $f(x)$ (for a detailed treatment of what is highlighted here we refer reader to [9]). The *mean* of x we denote by μ_x and the *variance* of x we denote by σ_x^2 (or equivalently by $V[x]$). Mean and variance of x are given by the following expressions:

$$
\mu_x = E[x] = \int_{-\infty}^{\infty} x f(x)dx\,, \tag{D1}
$$

$$
\begin{aligned}
\sigma_x^2 = V[x] &= E[(x-E[x])^2]\\
&= \int_{-\infty}^{\infty}(x-\mu_x)^2 f(x)dx\,,
\end{aligned}
\tag{D2}
$$

where $E[x]$ stands for the *expectation value* of a random variable x. We denote by x_i the measured random observable x in the i-th event and by $(w_x)_i$ the observable's weight in that event. Even if the p.d.f. $f(x)$ is completely unknown, we can still use measured values x_i to estimate mean (D1) and variance (D2) of a random variable x. In particular, the unbiased estimator [18] for the variance

σ_x^2 we denote by s_x^2 and it is given by

$$s_x^2 \equiv \left[\frac{\sum_{i=1}^N (w_x)_i (x_i - \langle x\rangle)^2}{\sum_{i=1}^N (w_x)_i}\right] \times \left[\frac{1}{1 - \frac{\sum_{i=1}^N (w_x)_i^2}{\left[\sum_{i=1}^N (w_x)_i\right]^2}}\right], \tag{D3}$$

where we have introduced also $\langle x\rangle$ as the unbiased estimator for mean μ_x,

$$\langle x\rangle \equiv \frac{\sum_{i=1}^N (w_x)_i x_i}{\sum_{i=1}^N (w_x)_i}. \tag{D4}$$

In above two equations N is the number of independent measurements, which in our context corresponds to the number of events.

Since the sample mean, $\langle x\rangle$, is an unbiased estimator for the mean of x, μ_x, we will report the final results and the statistical errors as

$$\langle x\rangle \pm V[\langle x\rangle]^{1/2}. \tag{D5}$$

One can easily show that the variance of the sample mean, $V[\langle x\rangle]$, can be written as

$$V[\langle x\rangle] = \frac{\sum_{i=1}^N (w_x)_i^2}{\left[\sum_{i=1}^N (w_x)_i\right]^2} V[x], \tag{D6}$$

i.e.

$$V[\langle x\rangle] = \frac{\sum_{i=1}^N (w_x)_i^2}{\left[\sum_{i=1}^N (w_x)_i\right]^2} \sigma_x^2. \tag{D7}$$

Taking into account the unbiased estimator s_x^2 for the variance σ_x^2, we can now write down the expression we will use to report the final results and statistical errors of a random variable x:

$$\langle x\rangle \pm \frac{\sqrt{\sum_{i=1}^N (w_x)_i^2}}{\sum_{i=1}^N (w_x)_i} s_x, \tag{D8}$$

where $\langle x\rangle$ is given by Eq. (D4) and s_x by Eq. (D3).

Consider now the more general case when we deal with two random variables, x and y, and some other random variable h which is a function of x and y. Than the mean of $h(x,y)$, μ_h, is to first order given by

$$\mu_h \equiv E[h(x,y)] \approx h(\mu_x, \mu_y), \tag{D9}$$

and the variance of h, σ_h^2 (or equivalently $V[h]$), is to first order given by:

$$\begin{aligned}\sigma_h^2 = V[h] &\equiv E[h^2(x,y)] - E[h(x,y)]^2 \\ &\approx \left[\left(\frac{\partial h}{\partial x}\right)\Big|_{x=\mu_x, y=\mu_y}\right]^2 \sigma_x^2 \\ &+ \left[\left(\frac{\partial h}{\partial y}\right)\Big|_{x=\mu_x, y=\mu_y}\right]^2 \sigma_y^2 \\ &+ 2\left(\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\right)\Big|_{x=\mu_x, y=\mu_y} V_{xy},\end{aligned} \tag{D10}$$

where V_{xy} is *covariance* of two random variables x and y,

$$\begin{aligned}V_{xy} &\equiv E[(x-\mu_x)(y-\mu_y)] \\ &= E[xy] - E[x]E[y] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy - \mu_x\mu_y.\end{aligned} \tag{D11}$$

If we measure in the i-th event two observables x_i and y_i, whose weights are $(w_x)_i$ and $(w_y)_i$, than the unbiased estimator $\mathrm{Cov}(x,y)$ for their covariance V_{xy} is given by

$$\mathrm{Cov}(x,y) = \frac{\frac{\sum_{i=1}^N (w_x)_i (w_y)_i x_i y_i}{\sum_{i=1}^N (w_x)_i (w_y)_i} - \frac{\sum_{i=1}^N (w_x)_i x_i}{\sum_{i=1}^N (w_x)_i}\frac{\sum_{j=1}^N (w_y)_j y_j}{\sum_{j=1}^N (w_y)_j}}{1 - \frac{\sum_{i=1}^N (w_x)_i (w_y)_i}{\sum_{i=1}^N (w_x)_i \sum_{j=1}^N (w_y)_j}}. \tag{D12}$$

We will report the final result and statistical error for $h(x,y)$ as

$$\langle h\rangle \pm s_{\langle h\rangle}, \tag{D13}$$

where $\langle h\rangle$ is the unbiased estimator for the mean of $h(x,y)$ and it is to first order given by

$$\langle h\rangle = h(\langle x\rangle, \langle y\rangle), \tag{D14}$$

while $s_{\langle h\rangle}^2$ is the unbiased estimator for the variance $V[\langle h\rangle]$. The variance $V[\langle h\rangle]$ can be obtained to first order straightforwardly from Eq. (D10):

$$\begin{aligned}V[\langle h\rangle] &\approx \left[\left(\frac{\partial h}{\partial x}\right)\Big|_{x=\mu_x, y=\mu_y}\right]^2 V[\langle x\rangle] \\ &+ \left[\left(\frac{\partial h}{\partial y}\right)\Big|_{x=\mu_x, y=\mu_y}\right]^2 V[\langle y\rangle] \\ &+ 2\left(\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\right)\Big|_{x=\mu_x, y=\mu_y} V_{\langle x\rangle\langle y\rangle}.\end{aligned} \tag{D15}$$

One can easily show that

$$V_{\langle x\rangle\langle y\rangle} = \frac{\sum_{i=1}^N (w_x)_i (w_y)_i}{\sum_{i=1}^N (w_x)_i \sum_{j=1}^N (w_y)_j} V_{xy}. \tag{D16}$$

Than the unbiased estimator $s_{\langle h\rangle}^2$ for variance $V[\langle h\rangle]$ is

$$\begin{aligned}s_{\langle h\rangle}^2 &\approx \left[\left(\frac{\partial h}{\partial x}\right)\Big|_{x=\langle x\rangle, y=\langle y\rangle}\right]^2 \frac{\sum_{i=1}^N (w_x)_i^2}{\left[\sum_{i=1}^N (w_x)_i\right]^2} s_x^2 \\ &+ \left[\left(\frac{\partial h}{\partial y}\right)\Big|_{x=\langle x\rangle, y=\langle y\rangle}\right]^2 \frac{\sum_{i=1}^N (w_y)_i^2}{\left[\sum_{i=1}^N (w_y)_i\right]^2} s_y^2 \\ &+ 2\left(\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\right)\Big|_{x=\langle x\rangle, y=\langle y\rangle} \\ &\times \frac{\sum_{i=1}^N (w_x)_i (w_y)_i}{\sum_{i=1}^N (w_x)_i \sum_{j=1}^N (w_y)_j} \mathrm{Cov}(x,y).\end{aligned} \tag{D17}$$

These formulas can be trivially generalized for the case of more than two random variables. We will use these results to report final results and statistical errors for the flow estimates from Q-cumulants in subsequent sections.

2. Statistical errors for reference flow estimates

We start by identifying random variable x in our analysis. We will denote its event-by-event measured (sampled) value by x_i and its weight by $(w_x)_i$.

In what follows we will treat the average multi-particle correlations, $\langle 2\rangle$ and $\langle 4\rangle$, as the measured observables event-by-event. Their values event-by-event are obtained from the formulas (29) and (32), respectively. Because of this we can calculate their final averages (i.e. the unbiased estimators for their mean values), $\langle\langle 2\rangle\rangle$, and $\langle\langle 4\rangle\rangle$, and also the unbiased estimators for their variances, $s^2_{\langle 2\rangle}$ and $s^2_{\langle 4\rangle}$, straight from the data by making use of the definitions (D4) and (D3), respectively.

Having calculated this straight from the data, we will report the final results and statistical errors for the average multiparticle azimuthal correlations in the following way:

$$\begin{aligned}\langle\langle 2\rangle\rangle &\pm \frac{\sqrt{\sum_{i=1}^N (w_{\langle 2\rangle})_i^2}}{\sum_{i=1}^N (w_{\langle 2\rangle})_i}\, s_{\langle 2\rangle}\,,\\ \langle\langle 4\rangle\rangle &\pm \frac{\sqrt{\sum_{i=1}^N (w_{\langle 4\rangle})_i^2}}{\sum_{i=1}^N (w_{\langle 4\rangle})_i}\, s_{\langle 4\rangle}\,. \end{aligned} \tag{D18}$$

On the other hand, we will report the final results and statistical errors of the flow estimates from Q-cumulants by taking into account their functional dependence on multi-particle correlations. In accordance with notation introduced in previous section (see Eq. (D13)) we will report the final results and statistical errors of reference flow estimates as follows:

$$\begin{aligned}\langle v_n\{2\}\rangle &\pm s_{\langle v_n\{2\}\rangle}\,,\\ \langle v_n\{4\}\rangle &\pm s_{\langle v_n\{4\}\rangle}\,.\end{aligned} \tag{D19}$$

Unbiased estimators for the variances of the sample mean of reference flow estimates, $s_{\langle v_n\{2\}\rangle}$ and $s_{\langle v_n\{4\}\rangle}$, introduced in Eq. (D19), can be straightforwardly expressed in terms of unbiased estimators for the variances of the sample mean of multiparticle correlations, $s_{\langle 2\rangle}$ and $s_{\langle 4\rangle}$, which as already indicated can be obtained straight from the data—these expressions will follow shortly.

But before proceeding further the very important thing to note, however, is that the different order average multi-particle correlations measured event-by-event *are not independent quantities.* Due to this, we will also need the unbiased estimators for their covariance $\mathrm{Cov}(\langle 2\rangle,\langle 4\rangle)$, which can also be calculated straight from the data by making use of Eq. (D12).

a. $2^{\rm nd}$ *order*

When it comes to $2^{\rm nd}$ order reference flow estimate we use the fact that

$$v_n\{2\} = \sqrt{\langle 2\rangle}\,, \tag{D20}$$

so that we have to first order

$$\langle v_n\{2\}\rangle \approx \sqrt{\langle\langle 2\rangle\rangle}\,. \tag{D21}$$

By restricting result in Eq.(D17) to functional dependence on one variable it follows

$$s^2_{\langle v_n\{2\}\rangle} = \frac{1}{4\,\langle\langle 2\rangle\rangle}\frac{\sum_{i=1}^N (w_{\langle 2\rangle})_i^2}{\left[\sum_{i=1}^N (w_{\langle 2\rangle})_i\right]^2}\, s^2_{\langle 2\rangle}\,, \tag{D22}$$

i.e.

$$s_{\langle v_n\{2\}\rangle} = \frac{1}{2\sqrt{\langle\langle 2\rangle\rangle}}\frac{\sqrt{\sum_{i=1}^N (w_{\langle 2\rangle})_i^2}}{\sum_{i=1}^N (w_{\langle 2\rangle})_i}\, s_{\langle 2\rangle}\,. \tag{D23}$$

In the next section we provide the formulas for the fourth order.

b. $4^{\rm th}$ *order*

We start from

$$v_n\{4\} = \sqrt[4]{2\cdot\langle 2\rangle^2 - \langle 4\rangle}\,, \tag{D24}$$

which gives to first order

$$\langle v_n\{4\}\rangle \approx \sqrt[4]{2\cdot\langle\langle 2\rangle\rangle^2 - \langle\langle 4\rangle\rangle}\,. \tag{D25}$$

By applying Eq. (D17) to equation (D24) we have

$$\begin{aligned} s^2_{\langle v_n\{4\}\rangle} = {} & \frac{1}{\left[2\cdot\langle\langle 2\rangle\rangle^2 - \langle\langle 4\rangle\rangle\right]^{3/2}}\\ & \times\left[\langle\langle 2\rangle\rangle^2\ \frac{\sum_{i=1}^N (w_{\langle 2\rangle})_i^2}{\left[\sum_{i=1}^N (w_{\langle 2\rangle})_i\right]^2}\, s^2_{\langle 2\rangle}\right.\\ & + \frac{1}{16}\cdot\frac{\sum_{i=1}^N (w_{\langle 4\rangle})_i^2}{\left[\sum_{i=1}^N (w_{\langle 4\rangle})_i\right]^2}\, s^2_{\langle 4\rangle}\\ & \left. -\frac{1}{2}\cdot\langle\langle 2\rangle\rangle\frac{\sum_{i=1}^N (w_{\langle 2\rangle})_i (w_{\langle 4\rangle})_i}{\sum_{i=1}^N (w_{\langle 2\rangle})_i \sum_{j=1}^N (w_{\langle 4\rangle})_j}\,\mathrm{Cov}(\langle 2\rangle,\langle 4\rangle)\right]. \end{aligned} \tag{D26}$$

3. Statistical errors for differential flow estimates

We will treat statistical errors of differential flow estimates by following the full analogy with the treatment of statistical errors of reference flow estimates presented in the previous section.

As a measured random observable event-by-event we identify the average reduced multi-particle correlations, $\langle 2'\rangle$ and $\langle 4'\rangle$. Their values for the bin of interest in particular event are given by Eq. (60) and (66), respectively. By following an analogy with the previous section, we will write the final results for the average reduced correlations and report their statistical errors as

$$\begin{aligned} \langle\langle 2'\rangle\rangle &\pm \frac{\sqrt{\sum_{i=1}^{N}(w_{\langle 2'\rangle})_i^2}}{\sum_{i=1}^{N}(w_{\langle 2'\rangle})_i}\, s_{\langle 2'\rangle}\,, \\ \langle\langle 4'\rangle\rangle &\pm \frac{\sqrt{\sum_{i=1}^{N}(w_{\langle 4'\rangle})_i^2}}{\sum_{i=1}^{N}(w_{\langle 4'\rangle})_i}\, s_{\langle 4'\rangle}\,. \end{aligned} \tag{D27}$$

All quantities in above two equations can be obtained straight from the data. Having this in mind we will report the final results and statistical errors for the differential flow estimates by taking into account their functional dependence on the multiparticle correlations and propagating the statistical error from this dependence. In accordance with notation introduced in previous section (see Eq. (D13)) we will report the final results and statistical errors of differential flow estimates:

$$\begin{aligned} \langle v_n'\{2\}\rangle &\pm s_{\langle v_n'\{2\}\rangle}\,, \\ \langle v_n'\{4\}\rangle &\pm s_{\langle v_n'\{4\}\rangle}\,. \end{aligned} \tag{D28}$$

As in the case of reference flow in previous section, the very important thing to note is that the different order average multi-particle correlations and the different order average reduced multi-particle correlations measured event-by-event *are not mutually independent quantities.* Hence, we also need the unbiased estimators for their covariances $\mathrm{Cov}(\langle 2\rangle,\langle 2'\rangle)$, $\mathrm{Cov}(\langle 2\rangle,\langle 4'\rangle)$, $\mathrm{Cov}(\langle 4\rangle,\langle 2'\rangle)$, $\mathrm{Cov}(\langle 4\rangle,\langle 4'\rangle)$ and $\mathrm{Cov}(\langle 2'\rangle,\langle 4'\rangle)$. These unbiased estimators for the covariances can also be obtained straight from the data by making use of Eq. (D12).

a. 2nd order

For the 2nd order differential flow estimate we have

$$v_n'\{2\} \equiv \frac{\langle 2'\rangle}{\langle 2\rangle^{1/2}}\,, \tag{D29}$$

which yields to first order

$$\langle v_n'\{2\}\rangle \approx \frac{\langle\langle 2'\rangle\rangle}{\langle\langle 2\rangle\rangle^{1/2}}\,. \tag{D30}$$

After plugging this functional dependence into Eq. (D17) we have

$$\begin{aligned} s^2_{\langle v_n'\{2\}\rangle} &= \frac{1}{4\cdot\langle\langle 2\rangle\rangle^3}\times\left[\langle\langle 2'\rangle\rangle^2\,\frac{\sum_{i=1}^{N}(w_{\langle 2\rangle})_i^2}{\left[\sum_{i=1}^{N}(w_{\langle 2\rangle})_i\right]^2}\, s^2_{\langle 2\rangle}\right. \\ &+ 4\cdot\langle\langle 2\rangle\rangle^2\,\frac{\sum_{i=1}^{N}(w_{\langle 2'\rangle})_i^2}{\left[\sum_{i=1}^{N}(w_{\langle 2'\rangle})_i\right]^2}\, s^2_{\langle 2'\rangle} - 4\cdot\langle\langle 2\rangle\rangle\,\langle\langle 2'\rangle\rangle \\ &\left.\times\ \frac{\sum_{i=1}^{N}(w_{\langle 2\rangle})_i(w_{\langle 2'\rangle})_i}{\sum_{i=1}^{N}(w_{\langle 2\rangle})_i\sum_{j=1}^{N}(w_{\langle 2'\rangle})_j}\,\mathrm{Cov}(\langle 2\rangle,\langle 2'\rangle)\right]. \end{aligned} \tag{D31}$$

In the next section we present the results for the 4th order differential flow.

b. 4th order

When it comes to 4th order differential flow estimate we start from

$$v_n'\{4\} \equiv \frac{2\cdot\langle 2\rangle\langle 2'\rangle - \langle 4'\rangle}{\left[2\cdot\langle 2\rangle^2 - \langle 4\rangle\right]^{3/4}}\,, \tag{D32}$$

which yields to leading order

$$\langle v_n'\{4\}\rangle \approx \frac{2\cdot\langle\langle 2\rangle\rangle\langle\langle 2'\rangle\rangle - \langle\langle 4'\rangle\rangle}{\left[2\cdot\langle\langle 2\rangle\rangle^2 - \langle\langle 4\rangle\rangle\right]^{3/4}}\,. \tag{D33}$$

We use again generalized version of Eq. (D17) valid for four random variables. It follows straightforwardly:

$$
\begin{aligned}
s^2_{\langle v'_n\{4\}\rangle} &= \frac{1}{\left[2\cdot\langle\langle 2\rangle\rangle^2-\langle\langle 4\rangle\rangle\right]^{\frac{7}{2}}}\times\\
&\Bigg\{\left[2\cdot\langle\langle 2\rangle\rangle^2\langle\langle 2'\rangle\rangle-3\cdot\langle\langle 2\rangle\rangle\langle\langle 4'\rangle\rangle+2\cdot\langle\langle 4\rangle\rangle\langle\langle 2'\rangle\rangle\right]^2\\
&\times\frac{\sum_{i=1}^N(w_{\langle 2\rangle})_i^2}{\left[\sum_{i=1}^N(w_{\langle 2\rangle})_i\right]^2}\,s^2_{\langle 2\rangle}\\
&+\frac{9}{16}\cdot[2\cdot\langle\langle 2\rangle\rangle\langle\langle 2'\rangle\rangle-\langle\langle 4'\rangle\rangle]^2\,\frac{\sum_{i=1}^N(w_{\langle 4\rangle})_i^2}{\left[\sum_{i=1}^N(w_{\langle 4\rangle})_i\right]^2}\,s^2_{\langle 4\rangle}\\
&+4\cdot\langle\langle 2\rangle\rangle^2\left[2\cdot\langle\langle 2\rangle\rangle^2-\langle\langle 4\rangle\rangle\right]^2\frac{\sum_{i=1}^N(w_{\langle 2'\rangle})_i^2}{\left[\sum_{i=1}^N(w_{\langle 2'\rangle})_i\right]^2}\,s^2_{\langle 2'\rangle}\\
&+\left[2\cdot\langle\langle 2\rangle\rangle^2-\langle\langle 4\rangle\rangle\right]^2\frac{\sum_{i=1}^N(w_{\langle 4'\rangle})_i^2}{\left[\sum_{i=1}^N(w_{\langle 4'\rangle})_i\right]^2}\,s^2_{\langle 4'\rangle}\\
&-\frac{3}{2}\cdot[2\cdot\langle\langle 2\rangle\rangle\langle\langle 2'\rangle\rangle-\langle\langle 4'\rangle\rangle]\\
&\times\left[2\cdot\langle\langle 2\rangle\rangle^2\langle\langle 2'\rangle\rangle-3\cdot\langle\langle 2\rangle\rangle\langle\langle 4'\rangle\rangle+2\cdot\langle\langle 4\rangle\rangle\langle\langle 2'\rangle\rangle\right]\\
&\times\frac{\sum_{i=1}^N(w_{\langle 2\rangle})_i(w_{\langle 4\rangle})_i}{\sum_{i=1}^N(w_{\langle 2\rangle})_i\sum_{j=1}^N(w_{\langle 4\rangle})_j}\,\mathrm{Cov}(\langle 2\rangle,\langle 4\rangle)\\
&-4\cdot\langle\langle 2\rangle\rangle\left[2\cdot\langle\langle 2\rangle\rangle^2-\langle\langle 4\rangle\rangle\right]\\
&\times\left[2\cdot\langle\langle 2\rangle\rangle^2\langle\langle 2'\rangle\rangle-3\cdot\langle\langle 2\rangle\rangle\langle\langle 4'\rangle\rangle+2\cdot\langle\langle 4\rangle\rangle\langle\langle 2'\rangle\rangle\right]\\
&\times\frac{\sum_{i=1}^N(w_{\langle 2\rangle})_i(w_{\langle 2'\rangle})_i}{\sum_{i=1}^N(w_{\langle 2\rangle})_i\sum_{j=1}^N(w_{\langle 2'\rangle})_j}\,\mathrm{Cov}(\langle 2\rangle,\langle 2'\rangle)\\
&+2\cdot\left[2\cdot\langle\langle 2\rangle\rangle^2-\langle\langle 4\rangle\rangle\right]\\
&\times\left[2\cdot\langle\langle 2\rangle\rangle^2\langle\langle 2'\rangle\rangle-3\cdot\langle\langle 2\rangle\rangle\langle\langle 4'\rangle\rangle+2\cdot\langle\langle 4\rangle\rangle\langle\langle 2'\rangle\rangle\right]\\
&\times\frac{\sum_{i=1}^N(w_{\langle 2\rangle})_i(w_{\langle 4'\rangle})_i}{\sum_{i=1}^N(w_{\langle 2\rangle})_i\sum_{j=1}^N(w_{\langle 4'\rangle})_j}\,\mathrm{Cov}(\langle 2\rangle,\langle 4'\rangle)\\
&+3\cdot\langle\langle 2\rangle\rangle\left[2\cdot\langle\langle 2\rangle\rangle^2-\langle\langle 4\rangle\rangle\right][2\cdot\langle\langle 2\rangle\rangle\langle\langle 2'\rangle\rangle-\langle\langle 4'\rangle\rangle]\\
&\times\frac{\sum_{i=1}^N(w_{\langle 4\rangle})_i(w_{\langle 2'\rangle})_i}{\sum_{i=1}^N(w_{\langle 4\rangle})_i\sum_{j=1}^N(w_{\langle 2'\rangle})_j}\,\mathrm{Cov}(\langle 4\rangle,\langle 2'\rangle)\\
&-\frac{3}{2}\cdot\left[2\cdot\langle\langle 2\rangle\rangle^2-\langle\langle 4\rangle\rangle\right][2\cdot\langle\langle 2\rangle\rangle\langle\langle 2'\rangle\rangle-\langle\langle 4'\rangle\rangle]\\
&\times\frac{\sum_{i=1}^N(w_{\langle 4\rangle})_i(w_{\langle 4'\rangle})_i}{\sum_{i=1}^N(w_{\langle 4\rangle})_i\sum_{j=1}^N(w_{\langle 4'\rangle})_j}\,\mathrm{Cov}(\langle 4\rangle,\langle 4'\rangle)\\
&-4\cdot\langle\langle 2\rangle\rangle\left[2\cdot\langle\langle 2\rangle\rangle^2-\langle\langle 4\rangle\rangle\right]^2\\
&\times\frac{\sum_{i=1}^N(w_{\langle 2'\rangle})_i(w_{\langle 4'\rangle})_i}{\sum_{i=1}^N(w_{\langle 2'\rangle})_i\sum_{j=1}^N(w_{\langle 4'\rangle})_j}\,\mathrm{Cov}(\langle 2'\rangle,\langle 4'\rangle)\Bigg\}.
\end{aligned}
\tag{D34}
$$

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[18] To treat statistical errors in this way was suggested by Evan Warren.

APPENDIX E: SPARE PLOTS

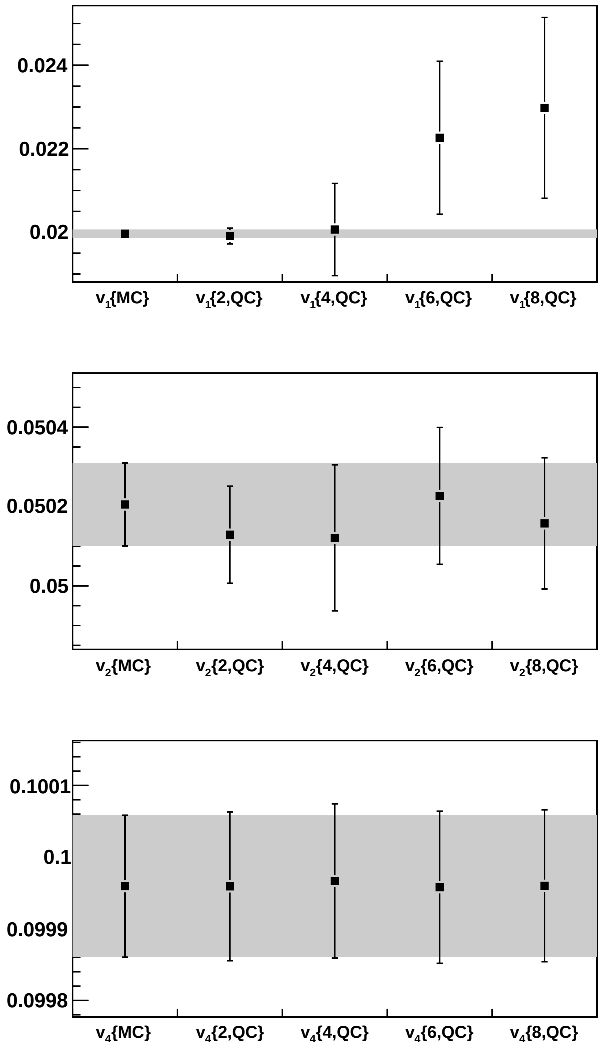

FIG. 17: Particle azimuthal angles were sampled from azimuthal distribution (1) characterized with a presence of three non-vanishing harmonics: $v_1 = 0.02$, $v_2 = 0.05$ and $v_4 = 0.10$. Estimates for harmonic v_1 are in the top plot, for harmonic v_2 in the middle plot and for the harmonic v_4 in the bottom plot. In each plot in the fist bin is Monte Carlo estimate to which estimates from $2^{\rm nd}$, $4^{\rm th}$, $6^{\rm th}$ and $8^{\rm th}$ Q-cumulant placed in subsequent bins are being compared. Each harmonic can be correctly estimated with Q-cumulants, the presence of other two harmonics is completely disentangled.

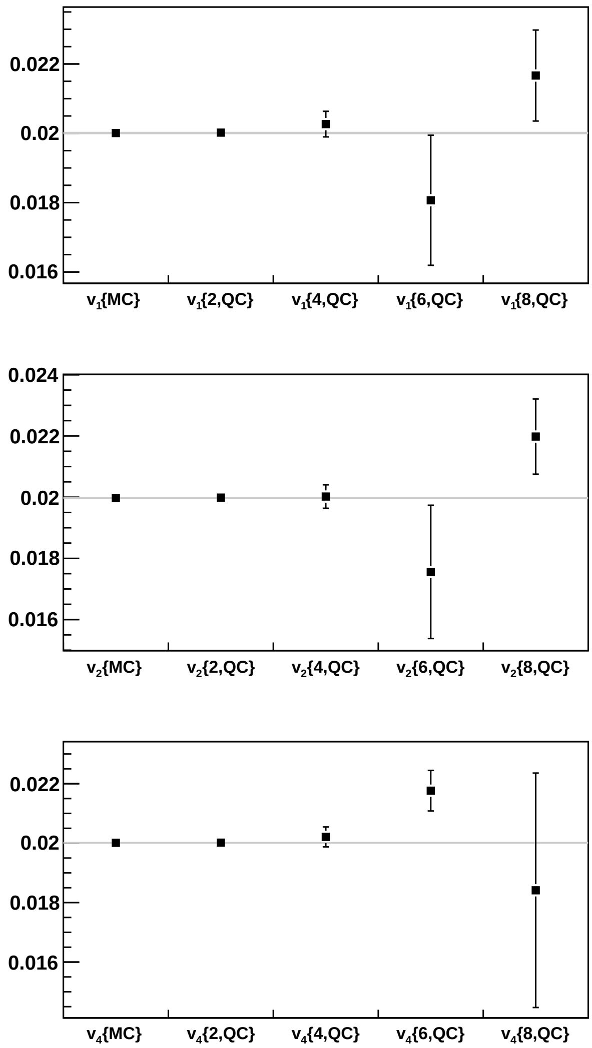

FIG. 18: Particle azimuthal angles were sampled from azimuthal distribution (1) characterized with a presence of one subdominant and two larger (order of magnitude) harmonics. On the top plot are estimates for subdominant harmonic $v_1 = 0.02$ in the presence of two large harmonics $v_2 = v_4 = 0.2$, in the middle plot are estimates for subdominant harmonic $v_2 = 0.02$ in the presence of two large harmonics $v_1 = v_4 = 0.2$ and on the bottom plot are estimates for subdominant harmonic $v_4 = 0.02$ in the presence of two large harmonics $v_1 = v_2 = 0.2$. Each harmonic can be correctly estimated with Q-cumulants, the presence of other two harmonics is completely disentangled.

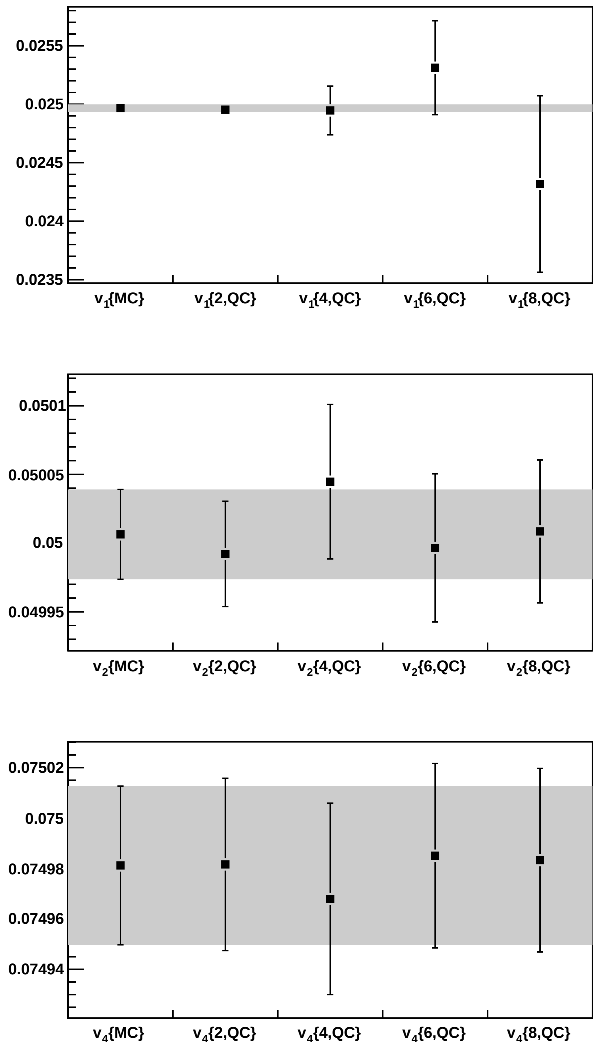

FIG. 19: Particle azimuthal angles were sampled from azimuthal distribution (1) characterized with a presence of three harmonics, namely $v_1 = 0.025$, $v_2 = 0.05$ and $v_4 = 0.075$. On the top plot are estimates for harmonic v_1, in the middle plot are estimates for harmonic v_2 and on the bottom plot are estimates for harmonic v_4. Each harmonic can be correctly estimated with Q-cumulants, the presence of other two harmonics is completely disentangled.

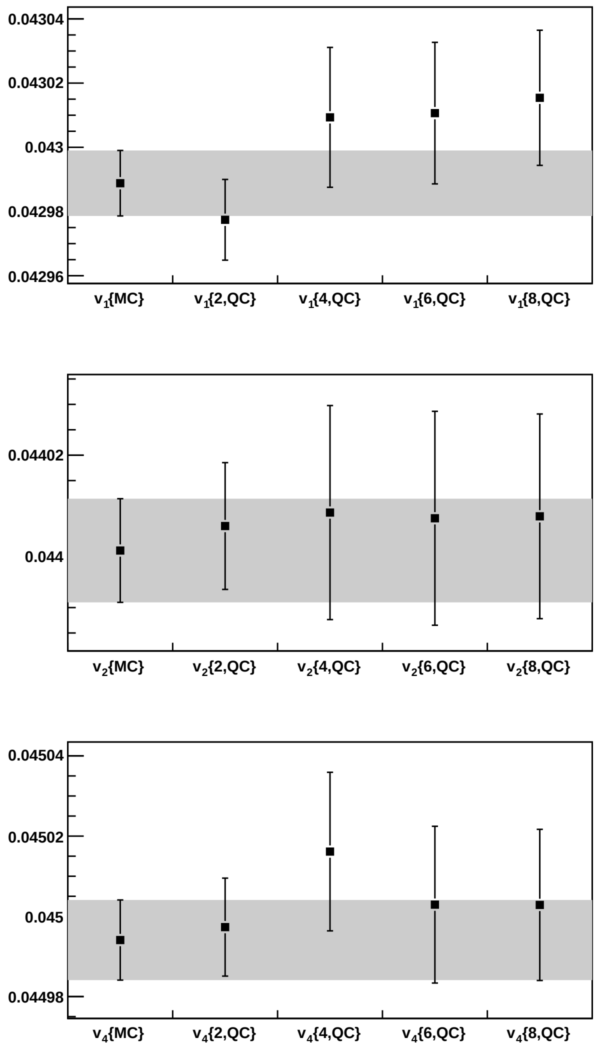

FIG. 20: Particle azimuthal angles were sampled from azimuthal distribution (1) characterized with a presence of three harmonics, namely $v_1 = 0.043$, $v_2 = 0.044$ and $v_4 = 0.045$. On the top plot are estimates for harmonic v_1, in the middle plot are estimates for harmonic v_2 and on the bottom plot are estimates for harmonic v_4. Each harmonic can be correctly estimated with Q-cumulants, the presence of other two harmonics is completely disentangled.

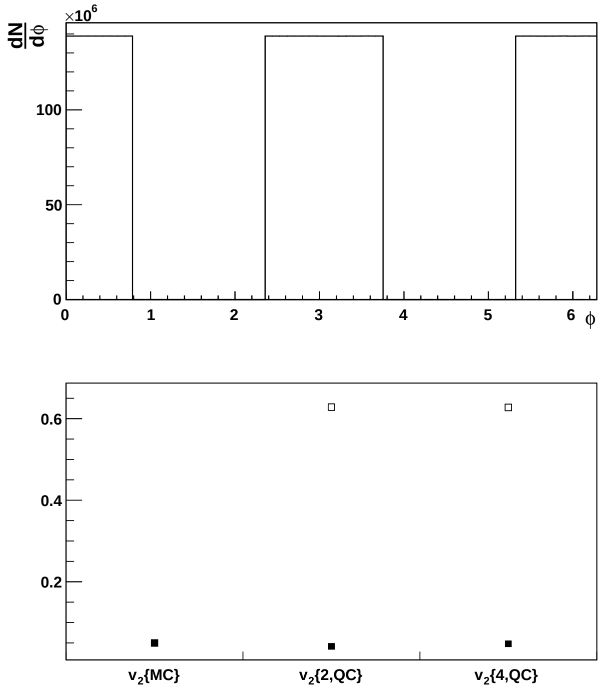

FIG. 21: Example for "PHENIX" like non-uniform acceptance when particles emitted in $45^o \leq \phi < 135^o$ and in $215^o \leq \phi < 305^o$ were blocked. Per event 1000 particles were sampled from azimuthal distribution (1) characterized with $v_2 = 0.05$ in total number of 10^7 events. Detector azimuthal profile is shown on the top plot. On the bottom plot with open markers are shown estimates from "isotropic" cumulants (defined in Eqs. (19) and (22)) and with closed markers estimates from "generalized" cumulants (Eqs. (C1) and (C6)). Bias due to detector defects is clearly huge (open markers) and has to be corrected for thoroughly (closed markers).